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Interpolation theory in sectorial Stieltjes classes and explicit system solutions

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Abstract

We introduce sectorial classes of matrix-valued Stieltjes functions in which we solve the bi-tangential interpolation problem of Nudelman and Ball–Gohberg–Rodman. We consider also a new type of solutions of Nevanlinna–Pick interpolation problems, so-called explicit system solutions generated by Brodskii–Livsic colligations, and find conditions on interpolation data of their existence and uniqueness. We point out the connections between sectorial Stieltjes classes and sectorial operators, and find out new properties of the classical Nevanlinna–Pick interpolation matrices (in the scalar case). We present in terms of interpolation data the exact formula for the angle of sectoriality of the main operator in the explicit system solution as well as the criterion for this operator to be extremal.The interpolation model for nonselfadjoint matrices is established. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

In this paper, we introduce and study sectorial classes of matrix-valued Stieltjes functions. These classes are the families $\mathscr{S}^{\cot\alpha}$ (with $\alpha \in (0, \pi/2)$) of $\mathbb{C}^{n\times n}$-valued functions analytic in the open upper half-plane $\mathbb{C}_+$ and such that the kernels

$$\frac{N(z) - N(w)^*}{z - w^*} \tag{1.1}$$

and

$$K_\alpha(w, z) = \frac{zN(z) - w^*N(w)^*}{z - w^*} - \cot\alpha\ N(w)^*N(z) \tag{1.2}$$

are positive there (in the sense of reproducing kernels; see Definition 2.1). The positivity of the kernel (1.1) means that N is in particular a Herglotz–Nevanlinna function.

For $0 < \alpha_1 \leqslant \alpha_2 < \pi/2$, we have

$$\mathscr{S}^{\cot\alpha_1} \subset \mathscr{S}^{\cot\alpha_2} \subset \mathscr{S}, \tag{1.3}$$

where $\mathscr{S}$ denotes the family of $\mathbb{C}^{n\times n}$-valued Stieltjes functions (which corresponds to the case $\alpha = \pi/2$), as follows from

$$K_{\alpha_1} = K_{\alpha_2} + (\cot\alpha_1 - \cot\alpha_2)N(w)^*N(z) \leqslant K_{\alpha_2} \leqslant K_{\pi/2}.$$

(Recall that for two functions $A(z, w)$ and $B(z, w)$ positive in a set E, the expression $A \leqslant B$ means that the difference $B - A$ is still positive on E.)

These classes appear, as we will prove in the sequel, in the theory of α-sectorial operators. When $\det(I_n - \cot\alpha\ N(z))$ is not identically vanishing, the positivity condition (1.2) is equivalent to the fact that

$$n(z) = zN(z)(I_n - \cot\alpha\ N(z))^{-1} \tag{1.4}$$

is a Herglotz–Nevanlinna function; see Lemma 2.3. It is therefore maybe no surprise that an important role is played in the sequel by the matrix of coefficients of the linear fractional transformation (1.4), i.e., by the matrix function

$$P(z) = \begin{pmatrix} zI_n & 0 \\ -\cot\alpha\ I_n & I_n \end{pmatrix}. \tag{1.5}$$

In fact, we will see that the automorphism of rational matrix-functions defined by

$$\mathscr{A}(W)(z) = P(z)W(z)P(z)^{-1}, \tag{1.6}$$

where W is a $\mathbb{C}^{2n\times 2n}$-valued rational function, has a fundamental role here. This automorphism preserves the class of J-unitary rational functions, where

$$J = \begin{pmatrix} 0 & -\mathrm{i}I_n \\ \mathrm{i}I_n & 0 \end{pmatrix}. \tag{1.7}$$

The paper consists of 11 sections beside Section 1 and is divided into two parts. Part I consists of Sections 2–7, and is devoted to the study of the bitangential interpolation problem of Nudelman and Ball–Gohberg–Rodman in the classes $\mathscr{S}^{\cot\alpha}$. In Part

II, we study the connections with the theory of α-sectorial operators.We consider a new type of solutions of Nevanlinna–Pick interpolation problems, so-called explicit system solutions generated by Brodskii–Livsic colligations, and find conditions on interpolation data of their existence and uniqueness. We point out the connections between sectorial Stieltjes classes and sectorial operators, and find out new properties of the classical Nevanlinna–Pick interpolation matrices (in the scalar case). We present in terms of interpolation data the exact formula for the angle of sectoriality of the main operator in the explicit system solution as well as the criterion for this operator to be extremal. The interpolation model for nonselfadjoint matrices is established.

The table of contents gives a more detailed idea of the structure of the paper.

Part I
Interpolation in the families $\mathscr{S}^{\cot\alpha}$

2. Some preliminaries

First we recall the definition of a positive function (see [10,29,32]). For a review of the role of positive functions in system theory, see e.g. [1].

Definition 2.1. Let Ω be some set. The $\mathbb{C}^{n\times n}$-valued function $K(z,w)$ defined for $z,w\in\Omega$ is called positive in Ω if it is hermitian: $K(z,w)=K(w,z)^*$ and if for all choices of $m\in\mathbb{N}$ and $w_1,\dots,w_m\in\mathbb{C}_+$ the $m\times m$ hermitian matrix with ℓ, j entry equal to $K(w_j,w_\ell)$ is nonnegative.

In the sequel we study interpolation in the classes $\mathscr{S}^{\cot\alpha}$. The general bitangential interpolation problem in the Stieltjes class was studied in [2], based on ideas developed in [3]. As in these papers, it is more convenient to consider the projective version of the classes $\mathscr{S}^{\cot\alpha}$.

Definition 2.2. A pair of $\mathbb{C}^{n\times n}$-valued functions N and M meromorphic in $\mathbb{C}_+$ is in the class $\mathscr{P}\mathscr{S}^{\cot\alpha}$ if

$$\operatorname{rank}\begin{pmatrix} N(z)\\ M(z)\end{pmatrix}=n \tag{2.1}$$

at all points of analyticity of N and M (at the possible exception of a zero set) and if the kernels

$$\frac{M(w)^*N(z)-N(w)^*M(z)}{z-w^*} \tag{2.2}$$

and

$$\frac{zM(w)^*N(z)-w^*N(w)^*M(z)}{z-w^*}-\cot\alpha\ N(w)^*N(z)$$

are positive in $\mathbb{C}_+$.

If $\det M(z) \not\equiv 0$, it follows in particular that the function $M^{-1}N$ is in the class $\mathscr{S}^{\cot\alpha}$. The pair is then said to have a *function representative*.

Lemma 2.3. *Let $N \in \mathscr{S}^{\cot\alpha}$ and assume*

$$\det(I_n - \cot\alpha\ N(z)) \not\equiv 0, \quad z \in \mathbb{C}_+. \tag{2.3}$$

Then, the function $n(z)$ defined by (1.4) *is in the Herglotz–Nevanlinna class.*

Proof. It suffices to write

$$\begin{aligned}
&\frac{zN(z) - z^*N(z)^*}{z - z^*} - \cot\alpha\ N(z)^*N(z) \\
&\quad = \frac{(I_n - \cot\alpha\ N(z)^*)(zN(z)) - z^*N(z)^*(I_n - \cot\alpha\ N(z))}{z - z^*} \\
&\quad = (I_n - \cot\alpha\ N(z))^* \\
&\qquad \times \frac{zN(z)(I_n - \cot\alpha N(z))^{-1} - (zN(z)(I_n - \cot\alpha N(z))^{-1})^*}{z - z^*} \\
&\qquad \times (I_n - \cot\alpha N(z)). \qquad \square
\end{aligned}$$

3. *J*-unitary rational functions and the automorphism (1.6)

In this section, we study the automorphism of rational matrix-functions defined by (1.6) and develop the analog of Section 3.2 of [3] in the present setting. The case $\cot\alpha = 0$ corresponds (up to the change of variable $z \mapsto 1/z$) to [3, Section 3.2]. Let W be a $\mathbb{C}^{2n\times 2n}$-valued rational function analytic at infinity. We set

$$W(z) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (zI_m - A)^{-1}(B_1 \ \ B_2)$$

to be a minimal realization of W, where the $D_{ij} \in \mathbb{C}^{n\times n}$, $C_i \in \mathbb{C}^{n\times m}$ and $B_i \in \mathbb{C}^{m\times n}$ for $i, j \in \{1, 2\}$. The function W is assumed moreover to be analytic at the origin. The block entries of $\mathscr{A}(W)(z) = P(z)W(z)P(z)^{-1}$ are given by

$$(\mathscr{A}(W))_{11}(z) = D_{11} + \cot\alpha\ D_{12} + C_1(zI_m - A)^{-1}(B_1 + \cot\alpha\ B_2), \tag{3.1}$$

$$(\mathscr{A}(W))_{12}(z) = zD_{12} + zC_1(zI_m - A)^{-1}B_2, \tag{3.2}$$

$$\begin{aligned}
(\mathscr{A}(W))_{21}(z) = \frac{1}{z}\Big(&D_{21} + \cot\alpha\ (D_{22} - D_{11}) - (\cot\alpha)^4\ D_{12} \\
&+ (C_2 - \cot\alpha\ C_1)(zI_m - A)^{-1}(B_1 + \cot\alpha\ B_2)\Big),
\end{aligned}$$

$$(\mathscr{A}(W))_{22}(z) = D_{22} - \cot\alpha\ D_{12} + (C_2 - \cot\alpha\ C_1)(zI_m - A)^{-1}B_2. \tag{3.3}$$

The analyticity of this function at infinity forces $D_{12} = 0$. Taking into account this condition, analyticity at the origin implies that

$$D_{21} + \cot\alpha\ (D_{22} - D_{11}) - (C_2 - \cot\alpha\ C_1)A^{-1}(B_1 + \cot\alpha\ B_2) = 0. \quad (3.4)$$

The functions under consideration are J-unitary on the real line. The theory of minimal realization and factorization of these functions was done in [8], from which the following result is taken (specialized to the present choice of signature matrix J).

Theorem 3.1. *Let W be a $\mathbb{C}^{2n\times 2n}$-valued rational function analytic at infinity and let $W(z) = D + C(zI_m - A)^{-1}B$ be a minimal realization of W. The function W takes J-unitary values on the real line if and only if the following conditions hold: D is J-unitary and there exists a hermitian and invertible matrix $\mathbb{P}$ such that*

$$A^*\mathbb{P} - \mathbb{P}A = C_1^*C_2 - C_2^*C_1, \quad (3.5)$$

$$B = \mathbb{P}^{-1}(C_2^* \ - C_1^*)D. \quad (3.6)$$

The hermitian matrix $\mathbb{P}$ is called the associated hermitian matrix to the given minimal realization.

Theorem 3.2. *Let W be a $\mathbb{C}^{2n\times 2n}$-valued rational function J-unitary on the real line and analytic at the origin and at infinity, and let a minimal realization of W be given by $W(z) = D + C(zI_m - A)^{-1}B$, with associated hermitian matrix $\mathbb{P}$. The function $\mathscr{A}(W)$ is analytic at the origin and at infinity if and only if $D_{21} = 0$ and*

$$D_{21} = \cot\alpha\ (D_{11} - D_{22}) + C_2^{\triangle}\mathbb{P}^{\triangle^{-1}}C_2^{\triangle *}D_{11}. \quad (3.7)$$

The matrix

$$\mathbb{P}^{\triangle} = \mathbb{P}A + C_1^*C_2^{\triangle} \quad (3.8)$$

is invertible and hermitian and a minimal realization of $\mathscr{A}(W)$ is given by

$$\mathscr{A}(W)(z) = \left(I_{2n} + \begin{pmatrix} C_1A \\ C_2^{\triangle}\end{pmatrix}(zI_m - A)^{-1}\mathbb{P}^{\triangle^{-1}}(C_2^{\triangle *} \quad - A^*C_1^*)\right) \times \begin{pmatrix} I_n & -C_1\mathbb{P}^{-1}C_1^* \\ 0 & I_n\end{pmatrix}\begin{pmatrix} D_{11} & 0 \\ 0 & D_{22}\end{pmatrix}, \quad (3.9)$$

where

$$C_2^{\triangle} = C_2 - \cot\alpha\ C_1. \quad (3.10)$$

Finally, a minimal realization of W is given by

$$W(z) = I_{2n} + \begin{pmatrix} C_1 \\ C_2\end{pmatrix}(zI_m - A)^{-1}\mathbb{P}^{-1}(C_2^* - C_1^*) \times \begin{pmatrix} I_n & 0 \\ C_2^{\triangle}\mathbb{P}^{\triangle^{-1}}C_2^{\triangle *} & I_n\end{pmatrix}\begin{pmatrix} D_{11} & 0 \\ \cot\alpha(D_{11} - D_{22}) & D_{22}\end{pmatrix}, \quad (3.11)$$

where D_{11} and D_{22} satisfy $D_{11}D_{22}^* = I_n$.

Proof. We first check that the matrix $\mathbb{P}^{\triangle}$ is hermitian and solves the Lyapunov equation

$$A^*\mathbb{P}^{\triangle} - \mathbb{P}^{\triangle}A = A^*C_1^*C_2^{\triangle} - C_2^{\triangle^*}C_1A. \tag{3.12}$$

To check that $\mathbb{P}^{\triangle}$ is hermitian we write (taking into account the Lyapunov equation (3.5)

$$\begin{aligned}\mathbb{P}^{\triangle} - \mathbb{P}^{\triangle^*} &= \mathbb{P}A + C_1^*C_2^{\triangle} - A^*\mathbb{P} - C_2^{\triangle^*}C_1\\ &= C_2^*C_1 - C_1^*C_2 + C_1^*C_2^{\triangle} - C_2^{\triangle^*}C_1\\ &= 0.\end{aligned}$$

We now check (3.12):

$$\begin{aligned}A^*\mathbb{P}^{\triangle} - \mathbb{P}^{\triangle}A &= A^*(\mathbb{P}A + C_1^*C_2^{\triangle}) - (A^*\mathbb{P} + C_2^{\triangle^*}C_1)A\\ &= A^*C_1^*C_2^{\triangle} - C_2^{\triangle^*}C_1A.\end{aligned}$$

Next, taking into account Eqs. (3.5) and (3.6), $D_{12} = 0$ and (3.4), we now compute the block entries of $\mathscr{A}(W)(\mathscr{A}(W)(\infty))^{-1}$. Taking into account (3.3) and (3.6) we have

$$\mathscr{A}(W)(\infty) = \begin{pmatrix} I_n & -C_1\mathbb{P}^{-1}C_1^* \\ 0 & I_n \end{pmatrix}\begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix}. \tag{3.13}$$

We first compute D_{21}: from (3.6) we have

$$B_1 + \cot\alpha\ B_2 = \mathbb{P}^{-1}((C_2^*D_{11} - \cot\alpha\ C_1^*D_{22}) - C_1^*D_{21}). \tag{3.14}$$

In conjunction with (3.4) we see that D_{21} solves the equation

$$\begin{aligned}&(I_n + C_2^{\triangle}A^{-1}\mathbb{P}^{-1}C_1^*)D_{21}\\ &\quad = \cot\alpha(D_{11} - D_{22}) + C_2^{\triangle}A^{-1}\mathbb{P}^{-1}(C_2^*D_{11} - \cot\alpha\ C_1^*D_{22}).\end{aligned}$$

Assume that $\det\mathbb{P}^{\triangle} \neq 0$. Then (and only then) is the matrix

$$(I_n + C_2^{\triangle}A^{-1}\mathbb{P}^{-1}C_1^*)$$

invertible and its inverse is given by

$$(I_n + C_2^{\triangle}A^{-1}\mathbb{P}^{-1}C_1^*)^{-1} = I_n - C_2^{\triangle}\mathbb{P}^{\triangle^{-1}}C_1^*.$$

We then have

$$\begin{aligned}D_{21} &= (I_n - C_2^{\triangle}\mathbb{P}^{\triangle^{-1}}C_1^*)(\cot\alpha(D_{11} - D_{22})\\ &\qquad + C_2^{\triangle}A^{-1}\mathbb{P}^{-1}(C_2^*D_{11} - \cot\alpha\ C_1^*D_{22}))\\ &= (\cot\alpha\ (D_{11} - D_{22}) + C_2^{\triangle}A^{-1}\mathbb{P}^{-1}(C_2^*D_{11} - \cot\alpha\ C_1^*D_{22}))\\ &\qquad - C_2^{\triangle}A^{-1}\mathbb{P}^{\triangle^{-1}}C_1^*\cot\alpha\ (D_{11} - D_{22})\end{aligned}$$

$$
\begin{aligned}
&-C_2^{\triangle}\mathbb{P}^{\triangle^{-1}}C_1^*C_2^{\triangle}A^{-1}\mathbb{P}^{-1}(C_2^*D_{11}-\cot\alpha\ C_1^*D_{22})\\
&=(\cot\alpha\ (D_{11}-D_{22})+C_2^{\triangle}A^{-1}\mathbb{P}^{-1}(C_2^*D_{11}-\cot\alpha\ C_1^*D_{22}))\\
&\quad -C_2^{\triangle}A^{-1}\mathbb{P}^{\triangle^{-1}}C_1^*\cot\alpha\ (D_{11}-D_{22})\\
&\quad -C_2^{\triangle}\mathbb{P}^{\triangle^{-1}}(\mathbb{P}^{\triangle}-\mathbb{P}A)A^{-1}\mathbb{P}^{-1}(C_2^*D_{11}-\cot\alpha\ C_1^*D_{22})\\
&=\cot\alpha\ (D_{11}-D_{22})+C_2^{\triangle}\mathbb{P}^{\triangle^{-1}}C_2^{\triangle^*}D_{11}.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
B_1+\cot\alpha\ B_2&=\mathbb{P}^{-1}(C_2^*D_{11}-\cot\alpha\ C_1^*D_{22}-C_1^*\cot\alpha\ D_{11}\\
&\qquad +C_1^*\cot\alpha\ D_{22}-C_1^*C_2^{\triangle}\mathbb{P}^{\triangle}C_2^{\triangle^*}D_{11}) \qquad (3.15)\\
&=\mathbb{P}^{-1}(I_m-C_1^*C_2^{\triangle}\mathbb{P}^{\triangle^{-1}})C_2^{\triangle^*}D_{11}\\
&=\mathbb{P}^{-1}(I_m-(\mathbb{P}^{\triangle}-\mathbb{P}A)\mathbb{P}^{\triangle^{-1}})C_2^{\triangle^*}D_{11}\\
&=A\mathbb{P}^{\triangle^{-1}}C_2^{\triangle^*}D_{11}. \qquad (3.16)
\end{aligned}
$$

We have moreover, taking into account (3.16),

$$
\begin{aligned}
(\mathscr{A}(W))_{11}(z)&=D_{11}+C_1(zI_m-A)^{-1}\mathbb{P}^{-1}(C_2^*D_{11}-C_1^*D_{21}-\cot\alpha\ C_1^*D_{22})\\
&=D_{11}+C_1(zI_m-A)^{-1}A\mathbb{P}^{\triangle^{-1}}C_2^{\triangle^*}D_{11}, \qquad (3.17)\\
(\mathscr{A}(W))_{12}(z)&=(-C_1\mathbb{P}^{-1}C_1^*-C_1A(zI_m-A)^{-1}\mathbb{P}^{-1}C_1^*)D_{22},\\
(\mathscr{A}(W))_{21}(z)&=C_2^{\triangle}(zI_m-A)^{-1}\mathbb{P}^{\triangle^{-1}}C_2^{\triangle^*}D_{11},\\
(\mathscr{A}(W))_{22}(z)&=(I-C_2^{\triangle}(zI_m-A)^{-1}\mathbb{P}^{-1}C_1^*)D_{22}. \qquad (3.18)
\end{aligned}
$$

We can now compute $\mathscr{A}(W)(z)(\mathscr{A}(W)(\infty))^{-1}$. In the computation of (3.20) and (3.22) we remark that the Lyapunov equation (3.5) and the definition (3.8) of $\mathbb{P}^{\triangle}$ imply that

$$
\begin{aligned}
\mathbb{P}^{\triangle}&=\mathbb{P}A+C_1^*(C_2-\cot\alpha\ C_1)\\
&=A^*\mathbb{P}+C_2^*C_1-C_1^*\cot\alpha\ C_1\\
&=A^*\mathbb{P}+C_2^{\triangle^*}C_1.
\end{aligned}
$$

The block entries of $\mathscr{A}(W)(z)(\mathscr{A}(W)(\infty))^{-1}$ are then

$$
(\mathscr{A}(W)(z)(\mathscr{A}(W)(\infty))^{-1})_{11}=I_n+C_1(zI_m-A)^{-1}A\mathbb{P}^{\triangle^{-1}}C_2^{\triangle^*}, \qquad (3.19)
$$

$$
\begin{aligned}
&(\mathscr{A}(W)(z)(\mathscr{A}(W)(\infty))^{-1})_{12}\\
&\quad=(\mathscr{A}(W))_{11}(z)D_{11}^{-1}+(\mathscr{A}(W))_{12}(z)D_{22}^{-1}C_1\mathbb{P}^{-1}C_1^*\\
&\quad=C_1\mathbb{P}^{-1}C_1^*+C_1A(zI-A)^{-1}\mathbb{P}^{\triangle^{-1}}C_2^{\triangle^*}C_1\mathbb{P}^{-1}C_1^*
\end{aligned}
$$

$$\begin{aligned}
&-C_1\mathbb{P}^{-1}C_1^* - C_1A(zI_m - A)^{-1}\mathbb{P}^{-1}C_1^*\\
&= C_1A(zI_m - A)^{-1}\mathbb{P}^{\triangle^{-1}}((C_2^{\triangle^*}C_1 - \mathbb{P}^{\triangle})\mathbb{P}^{-1}C_1^*\\
&= -C_1A(zI_m - A)^{-1}\mathbb{P}^{\triangle^{-1}}A^*C_1^*, \qquad (3.20)
\end{aligned}$$

$$(\mathscr{A}(W)(z)(\mathscr{A}(W)(\infty))^{-1})_{21} = C_2^{\triangle}(zI_m - A)^{-1}\mathbb{P}^{\triangle^{-1}}C_2^{\triangle^*} \qquad (3.21)$$

$$\begin{aligned}
&(\mathscr{A}(W)(z)(\mathscr{A}(W)(\infty))^{-1})_{22}\\
&= C_2^{\triangle}(zI_m - A)^{-1}\mathbb{P}^{\triangle^{-1}}C_2^{\triangle^*}C_1\mathbb{P}^{-1}C_1^*\\
&\quad + I_n - C_2^{\triangle}(zI_m - A)^{-1}\mathbb{P}^{-1}C_1^*\\
&= I_n + C_2^{\triangle}(zI_m - A)^{-1}\mathbb{P}^{\triangle^{-1}}(C_2^{\triangle^*}C_1 - \mathbb{P}^{\triangle})\mathbb{P}^{-1}C_1^*\\
&= I_n - C_2^{\triangle}(zI - A)^{-1}\mathbb{P}^{\triangle^{-1}}A^*C_1^*. \qquad \square \qquad (3.22)
\end{aligned}$$

The matrices

$$V = \begin{pmatrix} I_n & 0 \\ C_2^{\triangle}\mathbb{P}^{\triangle^{-1}}C_2^{\triangle^*} & I_n \end{pmatrix} \quad \text{and} \quad V^{\triangle} = \begin{pmatrix} I_n & -C_1\mathbb{P}^{-1}C_1^* \\ 0 & I_n \end{pmatrix} \qquad (3.23)$$

are J-unitary and will play a central role in the sequel.

4. Bitangential interpolation for Herglotz–Nevanlinna pairs

A pair (N, M) which satisfies the rank condition (2.1) is called a Herglotz–Nevanlinna pair; the general bitangential interpolation problem for Herglotz–Nevanlinna pairs was studied in [2,4–6]. We recall the definition of the bitangential interpolation problem for pairs as given in [2]. For an equivalent definition we refer to [4]. We will call a pair in $\mathscr{P}\mathscr{S}^{\cot\alpha}$ *canonical* if it is analytic in the open upper half-plane.

Problem 4.1. Given an ordered collection

$$\Omega = \left\{C_+,\ C_-,\ A_\pi,\ A_\zeta,\ B_+,\ B_-,\ \Gamma\right\} \qquad (4.1)$$

of seven matrices $C_+,\ C_- \in \mathbb{C}^{m\times n_\pi}$, $B_+,\ B_- \in \mathbb{C}^{n_\zeta\times m}$, $A_\pi \in \mathbb{C}^{n_\pi\times n_\pi}$, $A_\zeta \in \mathbb{C}^{n_\zeta\times n_\zeta}$ and $\Gamma \in \mathbb{C}^{n_\zeta\times n_\pi}$, the pair (C_-, A_π) being assumed observable,

$$\bigcap_{k=0}^{\infty} \ker\ C_-A_\pi^k = \{0\},$$

and the pair (A_ζ, B_+) being controllable,

$$\bigcup_{k=0}^{\infty} \operatorname{Im}\ A_\zeta^k B_+ = \mathbb{C}^{n_\zeta},$$

find all canonical pairs in $\mathscr{P}\mathscr{S}^{\cot\alpha}$ such that:

(1) The function

$$(zI_{n_\zeta} - A_\zeta)^{-1}(B_+N(z) + B_-M(z))$$

is analytic in $\mathbb{C}_+$.

(2) There is a $\mathbb{C}^{\ell \times n_\pi}$-valued function $H(z)$ meromorphic in $\mathbb{C}_+$ and such that

$$\begin{pmatrix} N(z) \\ M(z) \end{pmatrix} H(z) - \begin{pmatrix} C_+ \\ C_- \end{pmatrix} (zI_{n_\pi} - A_\pi)^{-1}$$

is analytic in $\mathbb{C}_+$.

(3) If $H(z)$ is as above, the function

$$\begin{aligned}(zI_{n_\zeta} - A_\zeta)^{-1}&(B_+ \quad B_-) \\ &\times \left(\begin{pmatrix} N(z) \\ M(z) \end{pmatrix} H(z) - \begin{pmatrix} C_+ \\ C_- \end{pmatrix} (zI_{n_\pi} - A_\pi)^{-1} \right) - (zI_{n_\pi} - A_\zeta)^{-1}\Gamma,\end{aligned}$$

is analytic in $\mathbb{C}_+$.

When $M(z) \equiv I_n$, we can take

$$H(z) = C_-(zI_{n_\pi} - A_\pi)^{-1}$$

to get the bitangential interpolation problem for functions defined in Section 5. This version of the bitangential interpolation problem for pairs in the class of Herglotz–Nevanlinna pairs is solved in [2]. The solution in the class $\mathscr{P}\mathscr{S}^{\cot\alpha}$ is presented in Theorem 4.2; the result is not needed in the sequel and the proof is omitted; we prove in Section 5 the corresponding result when all pairs which are solutions have a function representative (i.e., are such that $\det M(z) \not\equiv 0$).

Theorem 4.2. *Assume the matrices $\mathbb{P}$ and $\mathbb{Q}$ strictly positive, and build $\Theta(z)V$ as in Section 5. Then, the formula*

$$\begin{pmatrix} N(z) \\ M(z) \end{pmatrix} = \Theta(z)V \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \tag{4.2}$$

describes the set of pairs in $\mathscr{P}\mathscr{S}^{\cot\alpha}$ solution of the bitangential interpolation problem when $\left(\begin{smallmatrix} p(z) \\ q(z) \end{smallmatrix}\right)$ varies in all of $\mathscr{P}\mathscr{S}^{\cot\alpha}$.

It is of interest to know under which conditions all solutions of the bitangential interpolation problem in the set of Herglotz–Nevanlinna pairs are functions (i.e., are such that $\det M(z) \not\equiv 0$), or inverse of functions (i.e., such that $\det N(z) \not\equiv 0$). Such a study was done in [4, Corollary 3.5].

Proposition 4.3. *A necessary and sufficient condition for all solutions to the bitangential interpolation problem in the set of Herglotz–Nevanlinna pairs to be functions is that the mapping $(C_- \quad B_+^*)$ is surjective.*

5. Bitangential interpolation in $\mathscr{S}^{\cot\alpha}$

In this section, we follow the paper [2]. We consider the following two-sided residue problem.

Problem 5.1. Given an ordered collection

$$\Omega = \left\{C_+,\ C_-,\ A_\pi,\ A_\zeta,\ B_+,\ B_-,\ \Gamma\right\} \tag{5.1}$$

of seven matrices $C_+,\ C_- \in \mathbb{C}^{m\times n_\pi},\ B_+,\ B_- \in \mathbb{C}^{n_\zeta\times m},\ A_\pi \in \mathbb{C}^{n_\pi\times n_\pi},\ A_\zeta \in \mathbb{C}^{n_\zeta\times n_\zeta}$ and $\Gamma \in \mathbb{C}^{n_\zeta\times n_\pi}$, the pair (C_-, A_π) being assumed observable,

$$\bigcap_{k=0}^{\infty} \ker\ C_- A_\pi^k = \{0\},$$

and the pair (A_ζ, B_+) being controllable,

$$\bigcup_{k=0}^{\infty} \operatorname{Im}\ A_\zeta^k B_+ = \mathbb{C}^{n_\zeta},$$

find all functions in $\mathscr{S}^{\cot\alpha}$ such that

$$\sum_{\omega\in\mathbb{C}_+} \operatorname{Res}_{z=\omega}(zI_{n_\zeta} - A_\zeta)^{-1}B_+N(z) = -B_-, \tag{5.2}$$

$$\sum_{\omega\in\mathbb{C}_+} \operatorname{Res}_{z=\omega}N(z)C_-(zI_{n_\pi} - A_\pi)^{-1} = C_+, \tag{5.3}$$

$$\sum_{\omega\in\mathbb{C}_+} \operatorname{Res}_{z=\omega}(zI_{n_\zeta} - A_\zeta)^{-1}B_+N(z)C_-(zI_{n_\pi} - A_\pi)^{-1} = \Gamma. \tag{5.4}$$

The interpolation problem (5.2)–(5.4) was introduced by Ball et al. [11,12], and solved in various classes of functions (see [11]); we refer to these papers and to the monograph [20] for historical remarks on this problem, and mention the papers [30,31] of Nudelman. Condition (5.4) is added here in order to take into account the possible intersection of the spectra of the matrices A_ζ and A_π. We note that the matrix Γ satisfies the Sylvester equation

$$\Gamma A_\pi - A_\zeta\Gamma = B_+C_+ + B_-C_-. \tag{5.5}$$

The solution of this problem is given by the following theorem.

Theorem 5.2. *Let $\Omega = (C_+, C_-, A_\pi, A_\zeta, B_+, B_-, \Gamma)$ be an admissible interpolation data set such that the $(n_\pi + n_\zeta) \times (n_\pi + n_\zeta)$ matrix $\mathbb{P}$*

$$\mathbb{P} = \begin{pmatrix} S_1 & \Gamma^* \\ \Gamma & S_2 \end{pmatrix} \tag{5.6}$$

is positive definite, where S_1 *and* S_2 *are the* (*unique*) *solutions of the Lyapunov equations*

$$S_1 A_\pi - A_\pi^* S_1 = C_-^* C_+ - C_+^* C_-, \tag{5.7}$$

$$S_2 A_\zeta^* - A_\zeta S_2 = -B_+ B_-^* + B_- B_+^*, \tag{5.8}$$

and such that moreover the mapping $(C_- B_+^*)$ *is surjective. Then the bitangential interpolation problem* (5.2)–(5.4) *has all its solutions F in the Herglotz–Nevanlinna functions. The set of all Herglotz–Nevanlinna functions N solutions of* (5.2)–(5.4) *is described in the following way. Let*

$$\begin{aligned}\Theta(z) = I_{2n} &+ \begin{pmatrix} C_+ & -B_-^* \\ C_- & B_+^* \end{pmatrix} \begin{pmatrix} (zI_{n_\pi} - A_\pi)^{-1} & 0 \\ 0 & (zI_{n_\zeta} - A_\zeta^*)^{-1} \end{pmatrix} \\ &\times \mathbb{P}^{-1} \begin{pmatrix} C_-^* & -C_+^* \\ B_+ & B_- \end{pmatrix},\end{aligned} \tag{5.9}$$

let V be any J-unitary constant matrix and let $W = \Theta V$. *Then N is a Herglotz–Nevanlinna function satisfying the interpolation conditions* (5.2)–(5.4) *if and only if*

$$N(z) = (W_{11}(z)G(z) + W_{12}(z)H(z))(W_{21}(z)G(z) + W_{22}(z)H(z))^{-1}, \tag{5.10}$$

where (G, H) *is an arbitrary Herglotz–Nevanlinna pair.*

To solve the bitangential interpolation problem in the class $\mathscr{S}^{\cot\alpha}$, we show that the function $n(z)$ defined by (1.4) satisfies another two-sided interpolation problem.

Proposition 5.3. *Assume the function N is a solution of* (5.2)–(5.4). *Then, the function* $n(z)$ *defined by* (1.4) *solves*

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(zI_{n_\zeta} - A_\zeta)^{-1}(B_+ + \cot\alpha\ B_-)n(z) = -A_\zeta B_-, \tag{5.11}$$

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega} n(z)(C_- - \cot\alpha\ C_+)(zI_{n_\pi} - A_\pi)^{-1} = C_+ A_\pi, \tag{5.12}$$

$$\begin{aligned}&\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(zI_{n\zeta} - A_\zeta)^{-1}(B_+ + \cot\alpha\ B_-)n(z)(C_- - \cot\alpha\ C_+) \\ &\quad (zI_{n_\pi} - A_\pi)^{-1} = \Gamma A_\pi - B_-(C_- - \cot\alpha\ C_+).\end{aligned} \tag{5.13}$$

To prove this proposition we first need a preliminary lemma.

Lemma 5.4.

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(zI_{n_\zeta} - A_\zeta)^{-1} F_1(z) F_2(z)$$

$$= \sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(z - A_\zeta)^{-1} X_{F_1} F_2(z), \tag{5.14}$$

where

$$X_{F_1} = \sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(zI_{n_\zeta} - A_\zeta)^{-1} F_1(z) \tag{5.15}$$

and

$$\begin{aligned}&\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega} F_1(z)F_2(z)(zI_{n_\zeta} - A_\pi)^{-1}\\ &\quad= \sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega} F_1(z)Y_{F_2}(zI_{n_\pi} - A_\pi)^{-1},\end{aligned} \tag{5.16}$$

where

$$Y_{F_2} = \sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega} F_2(z)(zI_{n_\pi} - A_\pi)^{-1}. \tag{5.17}$$

Proof. We denote by γ a closed simple positively oriented contour in $\mathbb{C}_+$ which contains the spectrum of A_ζ. Note that

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(zI_{n_\zeta} - A_\zeta)^{-1}F(z) = \frac{1}{2\pi\mathrm{i}}\int_\gamma (zI_{n_\zeta} - A_\zeta)^{-1}F(z)\,\mathrm{d}z.$$

Thus we can write

$$\begin{aligned}&\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(zI_\zeta - A_\zeta)^{-1}X_{F_1}F_2(z)\\
&\quad= \frac{1}{(2\pi\mathrm{i})^2}\int_\gamma\int_\gamma (zI_\zeta - A_\zeta)^{-1}(wI - A_\zeta)^{-1}F_1(w)F_2(z)\,\mathrm{d}w\,\mathrm{d}z\\
&\quad= \frac{1}{(2\pi\mathrm{i})^2}\int_\gamma (zI_\zeta - A_\zeta)^{-1}\int_\gamma \frac{F_1(w)\,\mathrm{d}w}{w-z}F_2(z)\,\mathrm{d}z\\
&\qquad+\frac{1}{(2\pi\mathrm{i})^2}\int_\gamma (wI - A_\zeta)^{-1}F_1(w)\int_\gamma \frac{F_2(z)\,\mathrm{d}z}{z-w}\,\mathrm{d}w\\
&\quad= \frac{1}{2\pi\mathrm{i}}\int_\gamma (zI_\zeta - A_\zeta)^{-1}F_1(z)F_2(z)\,\mathrm{d}z\\
&\quad= \sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(zI_\zeta - A_\zeta)^{-1}F_1(z)F_2(z). \qquad \square\end{aligned}$$

Proof of Proposition 5.3. The functions n and N commute and are related by

$$n(z) - \cot\alpha\; N(z)n(z) = zN(z). \tag{5.18}$$

Hence,

$$
\begin{aligned}
&(zI_{n_\zeta} - A_\zeta)^{-1}B_+ n(z) - \cot\alpha\ (zI_{n_\zeta} - A_\zeta)^{-1}B_+ N(z)n(z)\\
&\quad = (zI_{n_\zeta} - A_\zeta)^{-1}B_+(zN(z)).
\end{aligned}
$$

Since

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(zI_{n_\zeta} - A_\zeta)^{-1}B_+N(z) = -B_-,$$

the first equality in the preceding lemma leads to

$$
\begin{aligned}
&\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(zI_{n_\zeta} - A_\zeta)^{-1}B_+n(z)\\
&\quad -\cot\alpha \sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}(zI_{n_\zeta} - A_\zeta)^{-1}(-B_-)n(z) = -A_\zeta B_-,
\end{aligned}
$$

and hence (5.11). The second equality is proved in much the same way and we turn to the last one: we multiply (5.18) on the left by $B_+^\triangle = B_+ + \cot\alpha\ B_-$ and by C_+ on the right and write $B_+^\triangle n = B_+^\triangle n + A_\zeta B_- - A_\zeta B_-$ and $NC_- = NC_- - C_+ + C_+$ to obtain the equality:

$$
\begin{aligned}
B_+^\triangle n(z)C_- = {} & \cot\alpha(B_+^\triangle n(z) + A_\zeta B_-)(N(z)C_- - C_+)\\
& + \cot\alpha(B_+^\triangle n(z) + A_\zeta B_-)C_+ - \cot\alpha A_\zeta B_-(N(z)C_- - C_+)\\
& - \cot\alpha A_\zeta B_-C_+ + zB_+N(z)C_- - z\cot\alpha\ B_-N(z)C_-,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
&B_+^\triangle n(z)C_- - \cot\alpha\ B_+^\triangle n(z)C_+\\
&\quad = \cot\alpha\ (B_+^\triangle n(z) + A_\zeta B_-)(N(z)C_- - C_+)\\
&\qquad +\cot\alpha\ A_\zeta B_-C_+ zB_+N(z)C_- + (zI_{n_\zeta} - A_\zeta)\cot\alpha\ B_-N(z)C_-.
\end{aligned}
\tag{5.19}
$$

Since the function

$$(zI_{n_\zeta} - A_\zeta)^{-1}(B_+^\triangle n(z) + A_\zeta B_-)(N(z)C_- - C_+)(zI_{n_\pi} - A_\pi)^{-1}$$

is analytic in $\mathbb{C}_+$, all its residues there are equal to 0. Thus, multiplying (5.19) on the left by $(zI_{n_\zeta} - A_\zeta)^{-1}$ and on the right by $(zI_{n_\pi} - A_\pi)^{-1}$ and applying the operator $\sum_{\omega\in\mathbb{C}_+}\mathrm{Res}$, we obtain the required equality. □

The set of functions n in $\mathscr{S}^{\cot\alpha}$ which satisfy the interpolation condition (5.11)–(5.13) is described as follows: set

$$\Gamma^\triangle = \Gamma A_\pi - B_-(C_- - \cot\alpha\ C_+), \tag{5.20}$$

and $S_1^\triangle$ and $S_2^\triangle$ be the solutions of the Lyapunov equations

$$S_1^\triangle A_\pi - A_\pi^* S_1^\triangle = C_-^{\triangle *}C_+^\triangle - C_+^{\triangle *}C_-^\triangle, \tag{5.21}$$

$$S_2^{\triangle} A_\zeta^* - A_\zeta S_2^{\triangle} = -B_+^{\triangle} B_-^{\triangle *} + B_-^{\triangle} B_+^{\triangle *}, \tag{5.22}$$

and

$$\mathbb{Q} = \begin{pmatrix} S_1^{\triangle} & \Gamma^{\triangle *} \\ \Gamma^{\triangle} & S_2^{\triangle} \end{pmatrix}. \tag{5.23}$$

Set

$$\Theta^{\triangle}(z) = I_{2n} + \begin{pmatrix} C_+ A_\pi & -B_-^* A_\zeta^* \\ C_- - \cot\alpha\ C_+ & B_+^* + \cot\alpha\ B_-^* \end{pmatrix} \tag{5.24}$$

$$\times \begin{pmatrix} (zI_{n_\pi} - A_\pi)^{-1} & 0 \\ 0 & (zI_{n_\zeta} - A_\zeta^*)^{-1} \end{pmatrix}$$

$$\times (\mathbb{P}^{\triangle})^{-1} \begin{pmatrix} C_-^* - \cot\alpha\ C_-^* & -A_\pi^* C_+^* \\ B_+ + \cot\alpha\ B_- & A_\zeta B_- \end{pmatrix}. \tag{5.25}$$

We are in the setting of formulas (3.11) and (3.9) and we can solve the bitangential interpolation problem in the class $\mathscr{S}^{\cot\alpha}$. We need to check that $\mathbb{P}$ and $\mathbb{P}^{\triangle}$ satisfy (3.8); we thus have to verify three identities, namely,

$$S_1^{\triangle} = S_1 A_\pi + C_+^* (C_- - \cot\alpha\ C_+), \tag{5.26}$$

$$S_2^{\triangle} = S_2 A_\zeta^* - B_- (B_+^* + \cot\alpha\ B_-^*), \tag{5.27}$$

$$\Gamma^{\triangle} = \Gamma A_\pi - B_- (C_- - \cot\alpha\ C_+).$$

The last one is nothing but (5.20). We turn to the first one: it is enough to show that the matrix

$$X = S_1 A_\pi + C_+^* (C_- - \cot\alpha\ C_+)$$

is also a solution of the Lyapunov equation (5.21): we first note that $X = X^*$: indeed, taking into account that S_1 is a solution of the Lyapunov equation (5.7), we have

$$\begin{aligned} X^* - X &= A_\pi^* S_1 + (C_-^* - \cot\alpha C_+^*) C_+ - S_1 A_\pi - C_+^* (C_- - \cot\alpha C_+) \\ &= -(C_-^* C_+ - C_+^* C_-) + (C_-^* - \cot\alpha C_+^*) C_+ - C_+^* (C_- - \cot\alpha C_+) \\ &= 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} X A_\pi - A_\pi^* X &= X^* A_\pi - A_\pi^* X \\ &= (A_\pi^* S_1 + (C_-^* - \cot\alpha\ C_+^*) C_+) A_\pi \\ &\quad - A_\pi^* (S_1 A_\pi + C_+^* (C_- - \cot\alpha\ C_+) \\ &= (C_-^* - \cot\alpha\ C_+^*) C_+ A_\pi - A_\pi^* C_+^* (C_- - \cot\alpha\ C_+) \\ &= C_-^{\triangle *} C_+^{\triangle} - C_+^{\triangle *} C_-^{\triangle}, \end{aligned}$$

so that $X = S_1^{\triangle}$ by uniqueness of the solution of the Lyapunov equation (5.21).

Theorem 5.5. *Assume that the matrices* $\mathbb{P}$ *and* $\mathbb{P}^{\triangle}$ *defined by* (5.6) *and* (5.23) *are strictly positive and that the mapping* $(C_- \quad B_+^*)$ *is surjective. Then, the bitangential interpolation problem* (5.1) *has solutions in* $\mathscr{S}^{\cot\alpha}$. *Let* Θ *be defined by the formula* (5.9) *and* $W = \Theta V$, *where* V *is the* J*-unitary constant matrix defined by*

$$V = \begin{pmatrix} I_n & 0 \\ Y\mathbb{P}^{\triangle -1}Y^* & I_n \end{pmatrix},$$

with

$$Y = (C_- - \cot\alpha\ C_+\ B_+^* + \cot\alpha\ B_-^*).$$

Then the linear fractional transformation

$$N(z) = (W_{11}(z)G(z) + W_{12}(z)H(z))(W_{21}(z)G(z) + W_{22}(z)H(z))^{-1}, \quad (5.28)$$

where (G, H) *is an arbitrary element of* $\mathscr{P}\mathscr{S}^{\cot\alpha}$, *describes all solutions of problem* (5.1) *in* $\mathscr{S}^{\cot\alpha}$.

We note that a similar strategy was used in [9] to solve the tangential trigonometric moment problem on an interval.

Corollary 5.6. *In* (5.28) *assume that* $\det H(z) \not\equiv 0$. *Then, the function* GH^{-1} *is a Stieltjes function and*

$$\begin{aligned} &\lim_{z\to\infty} N(z) \\ &\quad = \left(\lim_{z\to\infty}(G(z)H(z)^{-1})\right)\left(Y\mathbb{P}^{\triangle-1}Y\left(\lim_{z\to\infty}(G(z)H(z)^{-1})\right) + I_n\right)^{-1}. \end{aligned} \quad (5.29)$$

In Sections 6 and 7, we illustrate these results on examples.

6. Interpolation in the classes $\mathscr{S}^{\cot\alpha}$: an example

We address the following problem.

Problem 6.1. Given two sets of numbers $z_1, \ldots, z_p$ (with $z_j \neq z_k^*$) and $v_1, \ldots, v_p$, find all functions $N \in \mathscr{S}^{\cot\alpha}$ such that

$$N(z_j) = v_j, \quad j = 1, \ldots, p. \quad (6.1)$$

More generally, one has the tangential version.

Problem 6.2. Given p points $z_1, \ldots, z_p$ in the upper half-plane and $2p$ set of vectors $\xi_1, \ldots, \xi_p$ and $\eta_1, \ldots, \eta_p$ in $\mathbb{C}^n$, find all $N \in \mathscr{S}^{\cot\alpha}$ such that

$$N(z_j)\xi_j = \eta_j, \quad j = 1, \ldots, p. \quad (6.2)$$

Let $\mathbb{P}$ be the $p \times p$ hermitian matrix with i, j entry equal to

$$\mathbb{P}_{ij} = \frac{\xi_j^* \eta_i - \eta_j^* \xi_i}{z_i - z_j^*}. \tag{6.3}$$

A necessary condition for Problem 6.2 to have a solution in the class of Herglotz–Nevanlinna functions is that $\mathbb{P} \geqslant 0$. We will assume that $\mathbb{P} > 0$ and that

$$\ker(\xi_1^* \cdots \xi_p^*) = \{0\}. \tag{6.4}$$

Under these hypothesis, as recalled in Theorem 5.5, the set of solutions of Problem 6.2 in the set of Herglotz–Nevanlinna functions is described as follows: set

$$A = \operatorname{diag}(z_1, \ldots, z_p), \quad C_1 = (\eta_1\ \eta_2 \cdots \eta_p), \quad C_2 = (\xi_1\ \xi_2 \cdots \xi_p), \tag{6.5}$$

and

$$\Theta(z) = I_{2n} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (zI_p - A)^{-1} \mathbb{P}^{-1} (C_1^* C_2^*) \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{6.6}$$

Let $W = \Theta V$, with V be some fixed J-unitary matrix. A Herglotz–Nevanlinna function V satisfies the interpolation conditions (6.2) if and only if

$$N(z) = (W_{11}(z)G(z) + W_{12}(z)H(z))(W_{21}(z)G(z) + W_{22}(z)H(z))^{-1}, \tag{6.7}$$

where (G, H) is an arbitrary Herglotz–Nevanlinna pair. We will use the notation

$$N = T_\Theta(G, H)). \tag{6.8}$$

Lemma 6.3. *Assume that*

$$\ker(\xi_1 - \cot\alpha\eta_1 \cdots \xi_p - \cot\alpha\eta_p) = \{0\}, \tag{6.9}$$

and let N be a Herglotz–Nevanlinna function solution of the interpolation problem (6.2). *Then the function* $(I_n - \cot\alpha\ N(z))$ *is invertible in* $\mathbb{C}_+$ *and the function* (1.4) *is a Herglotz–Nevanlinna function.*

Proof. Let $w \in \mathbb{C}_+$ and $\xi \in \mathbb{C}^n$ be such that $(I - \cot\alpha N(z))\xi = 0$. The positivity of the function (1.2) implies that for every $j \in \{1, \ldots, p\}$, the matrix

$$\begin{pmatrix} \xi_j^* & 0 \\ 0 & \xi^* \end{pmatrix} \begin{pmatrix} K_\alpha(w, w) & K_\alpha(w_j, w) \\ K_\alpha(w, w_j) & K_\alpha(w_j, w_j) \end{pmatrix} \begin{pmatrix} \xi_j & 0 \\ 0 & \xi \end{pmatrix}$$

is nonnegative, i.e.,

$$\begin{pmatrix} 0 & \frac{-w^*\xi^* N(w)^*(\xi_j - \cot\alpha\ \eta_j)}{w^* - w_j} \\ \left(\frac{-w^*\xi^* N(w)^*(\xi_j - \cot\alpha\ \eta_j)}{w^* - w_j}\right)^* & \xi_j^* K_\alpha(w_j, w_j)\xi_j \end{pmatrix} \geqslant 0,$$

and hence

$$\xi^* N(w)^* (\xi_j - \cot\alpha \eta_j) = 0, \quad j = 1, \dots, n.$$

From the hypothesis (6.9), we get that $N(w)\xi = 0$ and hence $\xi = 0$ since we assume that $(I - \cot\alpha N(w))\xi = 0$. Hence a contradiction. □

It is readily seen that n satisfies the interpolation conditions

$$n(z_j)(\xi_j - \cot\alpha\ \eta_j) = w_j \eta_j, \quad j = 1, \dots, p. \tag{6.10}$$

This leads to another interpolation problem in the Herglotz–Nevanlinna class. A necessary condition for this problem to be solvable in this class is that $\mathbb{P}^{\triangle} \geqslant 0$, where

$$(\mathbb{P}^{\triangle})_{i,j} = \frac{\xi_j^{\triangle *} \eta_i^{\triangle} - \eta_j^{\triangle *} \xi_i^{\triangle}}{z_i - z_j^*} \tag{6.11}$$

with $\xi_j^{\triangle} = \xi_j - \cot\alpha\ \eta_j$ and $\eta_j^{\triangle} = w_j \eta_j$ for $j = 1, \dots, p$. Thus:

Proposition 6.4. *A sufficient condition for Problem* 6.2 *to have a solution in the class* $\mathscr{S}^{\cot\alpha}$ *is that both matrices* $\mathbb{P}$ *and* $\mathbb{P}^{\triangle}$ (*defined by* (6.3) *and* (6.11)) *be strictly positive and that the rank hypothesis* (6.4) *and* (6.9) *hold.*

The nonnegativity of the matrices $\mathbb{P}$ and $\mathbb{P}^{\triangle}$ is of course a direct consequence of the positivity of the kernels (9.5) and (9.6). It should be noted that, even in the case of Herglotz–Nevanlinna functions alone, the condition $\mathbb{P} \geqslant 0$ is not in general sufficient for a solution to exist.

With

$$C_1^{\triangle} = (\eta_1^{\triangle}\ \eta_2^{\triangle} \cdots \eta_p^{\triangle}) \quad \text{and} \quad C_2^{\triangle} = (\xi_1^{\triangle}\ \xi_2^{\triangle} \cdots \xi_p^{\triangle}), \tag{6.12}$$

define $\Theta^{\triangle}$ as Θ (formula (6.6)) with A as above but with $\mathbb{P}^{\triangle}$, $C_1^{\triangle}$ and $C_2^{\triangle}$ instead of $\mathbb{P}$, C_1 and C_2. Set $W^{\triangle} = \Theta^{\triangle} V^{\triangle}$, where $V^{\triangle}$ is any fixed J-unitary constant matrix. The function n is a strict Herglotz–Nevanlinna function which satisfies the interpolation conditions (6.10) if and only if

$$n = T_{\Theta^{\triangle}}(G^{\triangle}), \tag{6.13}$$

where $G^{\triangle}$ is an arbitrary strict Herglotz–Nevanlinna function. Taking into account that $n = T_P(N)$ we obtain

$$\begin{aligned} n &= T_P(N) \\ &= T_P(T_{\Theta V}(G)) \\ &= T_{P\Theta V P^{-1}}(T_P(G)). \end{aligned} \tag{6.14}$$

Assume we can find J-unitary constants V and $V^{\triangle}$ such that

$$P(z)\Theta(z)VP(z)^{-1} = \Theta^{\triangle}(z)V^{\triangle}. \tag{6.15}$$

Then, comparing (6.13) and (6.14), we obtain the following result:

Theorem 6.5. *Let V be a J-unitary constant such that* (6.15) *holds. The set of all solutions to Problem* 6.2 *in the class* $\mathscr{S}^{\cot\alpha}$ *is given by* $N = T_{\Theta V}(G, H))$, *where* (G, H) *is such that both* (G, H) *and* $T_P(G, H)$ *are Herglotz–Nevanlinna pairs.*

From the results in the previous section follows that the choices

$$V = \begin{pmatrix} I_n & 0 \\ C_2^{\triangle}\mathbb{P}^{\triangle-1}C_2^{\triangle *} & I_n \end{pmatrix}, \quad \text{and} \quad V^{\triangle} = \begin{pmatrix} I_n & -C_1\mathbb{P}^{-1}C_1^* \\ 0 & I_n \end{pmatrix}, \tag{6.16}$$

are adequate. Take A, C_1, C_2 and $\mathbb{P}$ defined in (6.5). Then, the Lyapunov equation (3.5) is satisfied, and therefore the matrix function Θ defined by (6.6) is J-unitary on the real line; it is in fact J-inner since $\mathbb{P}$ is not only nonsingular but assumed strictly positive. Note that $C_1^{\triangle}$ and $C_2^{\triangle}$ defined in (6.12) are in fact equal to

$$C_1^{\triangle} = C_1A, \quad C_2^{\triangle} = C_2 - \cot\alpha\; C_1, \tag{6.17}$$

and that $\mathbb{P}^{\triangle}$ (defined by (6.11)) solves the Lyapunov equation (3.12).

Claim 6.6. *Let* $\mathbb{P}$ *and* $\mathbb{P}^{\triangle}$ *be given by* (6.3) *and* (6.11), *respectively. Then,*

$$\mathbb{P}^{\triangle} = \mathbb{P}A + C_1^*C_2^{\triangle},$$

where $A = \operatorname{diag}(z_1, \ldots, z_p)$, $C_1 = (\eta_1\ \eta_2 \cdots \eta_p)$ *and*

$$C_2^{\triangle} = (\xi_1 - \cot\alpha\ \eta_1\ \xi_2 - \cot\alpha\ \eta_2 \cdots \xi_p - \cot\alpha\ \eta_p).$$

Proof. We have to prove that

$$\begin{aligned} &\frac{(\xi_j - \cot\alpha\ \eta_j)^* z_i\eta_i - z_j^*\eta_j^*(\xi_i - \cot\alpha\ \eta_i)}{z_i - z_j^*} \\ &\quad = \frac{\xi_j^*\eta_i - \eta_j^*\xi_i}{z_i - z_j^*} z_i + \eta_j^*(\xi_i - \cot\alpha\ \eta_i), \end{aligned}$$

but this is a straightforward computation. □

To conclude, it suffices to use formulas (3.9) and (3.11) with $D_{11} = D_{22} = I_n$, i.e., take V and $V^{\triangle}$ as in (3.23).

7. Bitangential interpolation: an example

In this section, we solve a simple case of the general bitangential interpolation problem $\mathscr{S}^{\cot\alpha}$.

Problem 7.1. Given $w \in \mathbb{C}_+$ and preassigned vectors $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{C}^n$ and number $\Gamma \in \mathbb{C}$, find all N such that

$$\xi_1^* N(w) = \eta_1^*, \tag{7.1}$$

$$N(w)\xi_2 = \eta_2, \tag{7.2}$$

$$\xi_1^* N(w)' \xi_2 = \Gamma. \tag{7.3}$$

It will be assumed that the mapping $(\xi_1 \ \xi_2)$ is surjective.

Note the compatibility condition

$$\eta_1^* \xi_2 = \xi_1^* \eta_2,$$

which follows from (7.1) and (7.2).

The solution of Problem 7.1 in the class of Herglotz–Nevanlinna functions was given above.

Theorem 7.2. *Let*

$$Q_1 = \frac{\xi_1^*\eta_1 - \eta_1^*\xi_1}{w^* - w}, \quad Q_2 = \frac{\xi_2^*\eta_2 - \eta_2^*\xi_2}{w - w^*}$$

and

$$\mathbb{P} = \begin{pmatrix} Q_2 & \Gamma^* \\ \Gamma & Q_1 \end{pmatrix}, \tag{7.4}$$

and assume that $\mathbb{P} > 0$. *The set of all solutions in the Herglotz–Nevanlinna class of the bitangential interpolation problem is described as follows*: *set*

$$\Theta(z) = I_{2n} + \begin{pmatrix} \eta_2 & \eta_1 \\ \xi_2 & \xi_1 \end{pmatrix} \begin{pmatrix} z - w & 0 \\ 0 & z - w^* \end{pmatrix}^{-1} \mathbb{P}^{-1} \begin{pmatrix} \xi_2^* & -\eta_2^* \\ \xi_1^* & -\eta_1^* \end{pmatrix}, \tag{7.5}$$

and let $W = \Theta V$, *where* V *is any J-unitary constant matrix. Then, N is a solution of Problem* 7.1 *if and only if*

$$N(z) = (W_{11}(z)G(z) + W_{12}(z)H(z))(W_{21}(z)G(z) + W_{22}(z)H(z))^{-1}, \tag{7.6}$$

where (G, H) *is an arbitrary Herglotz–Nevanlinna pair.*

Lemma 7.3. *Let N be a solution of the two sided Interpolation Problem* 7.1. *Then, we have that* $\det(I_n - \cot\alpha \ N(z)) \not\equiv 0$ *and the function* $n(z)$ *defined by* (1.4) *solves the problem*

$$(\xi_1^* - \cot\alpha \ \eta_1^*)n(w) = w\eta_1^*, \tag{7.7}$$

$$n(w)(\xi_2 - \cot\alpha \ \eta_2) = w\eta_2, \tag{7.8}$$

$$(\xi_1^* - \cot\alpha \ \eta_1^*)n(w)'(\xi_2 - \cot\alpha \ \eta_2) = w\Gamma + \eta_1^*(\xi_2 - \cot\alpha \ \eta_2). \tag{7.9}$$

Proof. To prove directly the invertibility of $(I_n - \cot\alpha \ N(z))$, we remark that N has a (unique) extension to the open lower half-plane $\mathbb{C}_-$ such that $K_\alpha(z, w)$ is still positive in $\mathbb{C}_+ \cup \mathbb{C}_-$, namely $N(z^*) = N(z)^*$. We then replace the condition $\xi_1^* N(w) = \eta_1^*$ by $N(w^*)\xi_1 = \eta_1$, and we are just in the setting of the previous section. We have $(I_n - \cot\alpha\ N(z))n(z) = zN(z)$. Multiplying both sides on the left by ξ_1^*, setting $w = z$ and taking into account that $\xi_1^* N(w) = \eta_1^*$ we obtain the first condition. The second is proved similarly. To prove the third, we differentiate $(I_n - \cot\alpha\ N(z))n(z) = zN(z)$ to obtain

$$(I_n - \cot\alpha\ N(z))n'(z) - \cot\alpha\ N'(z)n(z) = N(z) + zN'(z).$$

Replacing *n* by its expression in terms of *N*, we obtain

$$(I_n - \cot\alpha\ N(z))n'(z)(I - \cot\alpha\ N(z)) = zN'(z) + N(z)(I - \cot\alpha N(z)).$$

Multiplying on the left by ξ_1^* and on the right by ξ_2 and setting $z = w$, we obtain the required equality, thanks to (7.3). □

The set of strict Herglotz–Nevanlinna functions *n* which satisfy (7.7)–(7.9) can be described as follows: set

$$\Gamma^{\triangle} = w\Gamma + \eta_1^*(\xi_2 - \cot\alpha\ \eta_2), \tag{7.10}$$

$$\eta_1^{\triangle} = w^*\eta_1, \tag{7.11}$$

$$\eta_2^{\triangle} = w\eta_2, \tag{7.12}$$

and

$$\xi_j^{\triangle} = \xi_j - \cot\alpha\ \eta_j, \quad j = 1, 2, \tag{7.13}$$

$$Q_2^{\triangle} = \frac{\xi_2^{\triangle *}\eta_2^{\triangle} - \eta_2^{\triangle *}\xi_2^{\triangle}}{w - w^*}, \tag{7.14}$$

$$Q_1^{\triangle} = \frac{\xi_1^{\triangle *}\eta_1^{\triangle} - \eta_1^{\triangle *}\xi_1^{\triangle}}{w^* - w}, \tag{7.15}$$

and define $\mathbb{P}^{\triangle}$ as $\mathbb{P}$ in formula (7.4), but with $Q_1^{\triangle}, Q_2^{\triangle}, \Gamma^{\triangle}$ instead of $Q_1,\ Q_2$ and Γ:

$$\mathbb{P}^{\triangle} = \begin{pmatrix} Q_2^{\triangle} & \Gamma^{\triangle *} \\ \Gamma^{\triangle} & Q_1^{\triangle} \end{pmatrix}. \tag{7.16}$$

Then,

$$\Theta^{\triangle}(z) = I_{2n} + \begin{pmatrix} w\eta_2 & w^*\eta_1 \\ \xi_2 - \cot\alpha\ \eta_2 & \xi_1 - \cot\alpha\ \eta_1 \end{pmatrix} \begin{pmatrix} z - w & 0 \\ 0 & z - w^* \end{pmatrix}^{-1} \\ \times \mathbb{P}^{\triangle -1} \begin{pmatrix} \xi_2^* - \cot\alpha\ \eta_2^* & -w\eta_2 \\ \xi_1^* - \cot\alpha\ \eta_1^* & -w\eta_1 * \end{pmatrix}, \tag{7.17}$$

and let $V^{\triangle}$ be any *J*-unitary constant matrix. Then the linear fractional transformation $n = T_{\Theta^{\triangle} V^{\triangle}}(G^{\triangle})$ describes the set of all solutions *n*. To conclude, we need to find constants V and $V^{\triangle}$ which meet (6.15) for the present choices of Θ and $\Theta^{\triangle}$. For that purpose, we need to check that relationship (3.8) holds. This is a direct computation, which we check now.

Claim 7.4. *Let* $\mathbb{P}$ *and* $\mathbb{P}^{\triangle}$ *be given by* (7.4) *and* (7.16), *respectively. Then,*

$$\mathbb{P}^{\triangle} = \mathbb{P}A + C_1^* C_2^{\triangle},$$

where $A = \operatorname{diag}(w, w^*)$, $C_1 = (\eta_2\ \eta_1)$ *and* $C_2^{\triangle} = (\xi_2\ \xi_1) - \cot\alpha\ (\eta_2\ \eta_1)$.

Proof. There are three equalities to verify:

$$\Gamma^{\triangle} = w\Gamma + \eta_1^*(\xi_2 - \cot\alpha\ \eta_2), \tag{7.18}$$

and

$$Q_2^{\triangle} = wQ_2 + \eta_2^*(\xi_2 - \cot\alpha\ \eta_2), \tag{7.19}$$

$$Q_1^{\triangle} = w^* Q_1 + \eta_1^*(\xi_1 - \cot\alpha\ \eta_1). \tag{7.20}$$

The first one is just (7.10). We now check the third one. We thus have to verify that

$$\frac{\xi_1^{\triangle *}\eta_1^{\triangle} - \eta_1^{\triangle *}\xi_1^{\triangle}}{w^* - w} = w^* \frac{\xi_1^*\eta_1 - \eta_1^*\xi_1}{w^* - w} + \eta_1^*(\xi_1 - \cot\alpha\ \eta_1),$$

i.e., after multiplying both sides by $(w - w^*)$ and replacing $\xi_1^{\triangle}$ and $\eta_1^{\triangle}$ by the expressions (7.13) and (7.14), respectively,

$$\begin{aligned}(\xi_1^* - \cot\alpha\ \eta_1^*)w^*\eta_1 - w\eta_1^*(\xi_1 - \cot\alpha\ \eta_1)\\ = w^*(\xi_1^*\eta_1 - \eta_1^*\xi_1) + w^* - w)\eta_1^*(\xi_1 - \cot\alpha\ \eta_1),\end{aligned}$$

which is plain. Equality (7.19) is proved in the same way. □

Part II
α-Sectorial operators and interpolation in the classes $\mathscr{S}^{\cot\alpha}$

8. Characteristic operator-valued functions

This section and the following ones form the second part of the paper. In the present section, we give a number of definitions to set the operator framework of the paper.

Definition 8.1. A Brodskii–Livsic operator colligation (or Brodskii–Livsic time-invariant system) is an aggregate

$$\theta = \begin{pmatrix} T & K & J \\ \mathscr{G} & & E \end{pmatrix},$$

where $\mathscr{G}$ and E are Hilbert spaces, where T is a bounded linear operator from $\mathscr{G}$ into itself, K is a bounded linear operator from E into $\mathscr{G}$ and J is selfadjoint and unitary from E into itself, and furthermore,

$$\operatorname{Im} T = KJK^*.$$

Definition 8.2. Two Brodskii–Livsic time-invariant systems θ_1 and θ_2

$$\theta_1 = \begin{pmatrix} T_1 & K_1 & J \\ \mathscr{G}_1 & & E \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} T_2 & K_2 & J \\ \mathscr{G}_2 & & E \end{pmatrix}$$

are unitarily equivalent if there exists a unitary operator $U : \mathscr{G}_1 \to \mathscr{G}_2$ such that

$$UT_1 = T_2U, \quad UK_1 = K_2. \tag{8.1}$$

The operator-valued function

$$W_\theta(z) = I - 2\mathrm{i}\, K^*(T - zI)^{-1}KJ, \tag{8.2}$$

is called the *characteristic operator-valued function*, or the *transfer operator-valued function* of the Brodskii–Livsic colligation θ (or of the time-invariant Brodskii–Livsic system θ). The operator T is called the main operator of the system θ, and the space $\mathscr{G}$ is called the state space. The operator K is called the channel operator and J is called the direction operator. The case $J = I$ is of particular importance; the Brodskii–Livsic system is then called a *scattering system*.

The function

$$V_\theta(z) = \mathrm{i}(W_\theta(z) + I)^{-1}(W_\theta(z) - I)J = K^*(\operatorname{Re} T - zI)^{-1}K, \tag{8.3}$$

is an operator-valued Herglotz–Nevanlinna function, i.e., is analytic in the open upper half-plane $\mathbb{C}_+$ and has a positive imaginary part there. See [17,28]. As is well known and follows from the Herglotz representation formula for a function N analytic in the open upper half-plane and with positive imaginary part there

$$N(z) = A + zB + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{t^2 + 1} \right) \mathrm{d}\mu(t), \tag{8.4}$$

where

$$A = A^*, \quad B \geqslant 0, \quad \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t^2 + 1} < \infty$$

(see e.g. [17, pp. 23–25]), the condition

$$\frac{\operatorname{Im} V_\theta(z)}{\operatorname{Im} z} \geqslant 0$$

is equivalent to the seemingly stronger condition that the kernel

$$K_{V_\theta}(z, w) = \frac{V_\theta(z) - V_\theta(w)^*}{z - w^*} \tag{8.5}$$

is positive in the open upper half-plane. In fact, the Herglotz representation formula implies that the Herglotz–Nevanlinna function V_θ has an extension to the lower open half-plane given by $V_\theta(z^*)^* = V_\theta(z)$ and for which the kernel (8.5) is still positive in $\mathbb{C} \setminus \mathbb{R}$.

Not every Herglotz–Nevanlinna function admits a realization of the form (8.3). The Herglotz–Nevanlinna functions which do have such a representation are called *realizable* and were characterized in [16, Theorem 9, p. 68] (see also [14,15] where the results of [16] are announced). For other examples we also refer to [19,21–25,33]. An important problem is the study of realizability of Herglotz–Nevanlinna functions with a bounded α-sectorial operator.

For unitarily equivalent systems θ_1 and θ_2 we have $W_{\theta_1} = W_{\theta_2}$. The converse statement is true for prime systems: the Brodskii–Livsic system is called *prime* (or *minimal*; this latter terminology comes from system theory, see [13, pp. 19–21]) if it holds that

$$\bigvee_{\ell=0}^{\infty} \operatorname{ran} T^\ell K = \mathscr{G}.$$

This notion is important: two prime Brodskii–Livsic systems with same transfer function W_θ are unitarily equivalent.

Definition 8.3. A time-invariant Brodskii–Livsic system is called an interpolation system for the data $\{z_\ell, \ell = 1, \ldots, m\}$ and $\{v_\ell, \ell = 1, \ldots, m\}$ if it holds that

$$V_\theta(z_\ell) = v_\ell, \quad \ell = 1, \ldots, m, \tag{8.6}$$

where V_θ is defined by (8.3).

We will be interested in constructing, when possible, a Brodskii–Livsic scattering system with given interpolation data and with α-sectorial operator. We will also be interested in such systems with the state space $\mathscr{G}$ of minimal dimension. The following uniqueness result explains the importance of this last requirement.

Theorem 8.4. *Let* $\{z_\ell, \ell = 1, \ldots, m\}$ *and* $\{v_\ell, \ell = 1, \ldots, m\}$ *be an interpolation data* (*as in Definition* 8.3). *Let* θ_1 *and* θ_2 *be two prime Brodskii–Livsic interpolation scattering systems*

$$\theta_1 = \begin{pmatrix} T_1 & K_1 & I \\ \mathscr{G}_1 & & E \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} T_2 & K_2 & I \\ \mathscr{G}_2 & & E \end{pmatrix},$$

with finite dimensional E and finite dimensional state spaces $\mathscr{G}_1$ *and* $\mathscr{G}_2$ (*of dimensions* n_1 *and* n_2, *respectively*). *Assume furthermore that*

$$\bigvee_{\ell=1}^{m} \operatorname{ran}(T_1 - z_\ell I)^{-1} K_1 = \mathscr{G}_1, \quad \bigvee_{\ell=1}^{m} \operatorname{ran}(T_2 - z_\ell I)^{-1} K_2 = \mathscr{G}_2, \tag{8.7}$$

and that

$$|z_\ell| > \max(\|T_1\|, \|T_2\|), \quad \ell = 1, \ldots, m. \tag{8.8}$$

Then, the two systems are unitarily equivalent, $n_1 = n_2$, *and*

$$V_{\theta_1}(z) = V_{\theta_2}(z).$$

Proof. Condition (8.8) forces the points z_ℓ to be in the resolvent sets of the operators T_1 and T_2. Therefore the interpolation conditions $V_{\theta_1}(z_\ell) = V_{\theta_2}(z_\ell)$ for $\ell = 1, \ldots, m$ can be rewritten as

$$K_1^*(T_1 - z_\ell I)^{-1}K_1 = K_2^*(T_2 - z_\ell I)^{-1}K_2, \quad \ell = 1, \ldots, n, \tag{8.9}$$

that is, $W_{\theta_1}(z_\ell) = W_{\theta_2}(z_\ell),\ \ell = 1, \ldots, m$. We furthermore claim that it holds that

$$\begin{aligned}&\left\langle (T_1 - z_\ell I)^{-1}K_1\phi, (T_1 - z_j I)^{-1}K_1\phi\right\rangle_{\mathscr{G}_1}\\ &\quad= \left\langle (T_1 - z_\ell I)^{-1}K_1\phi, (T_2 - z_j I)^{-1}K_2\phi\right\rangle_{\mathscr{G}_2}\end{aligned} \tag{8.10}$$

for any $\phi \in E$ and all $\ell, j = 1, \ldots, m$. Indeed, taking into account that

$$\operatorname{Im} T_1 = K_1K_1^*, \quad \operatorname{Im} T_2 = K_2K_2^*,$$

we have that

$$\begin{aligned}&(T_1 - z_\ell I)^{-1} - (T_1^* - z_j^* I)^{-1}\\ &\quad= (T_1^* - z_j^* I)^{-1}\left((T_1^* - z_j^* I) - (T_1 - z_\ell I)\right)(T_1 - z_\ell)^{-1}\\ &\quad= -2\mathrm{i}(T_1^* - z_j^*)^{-1}K_1K_1^*(T_1 - z_\ell I)^{-1} +\\ &\qquad+ (z_\ell - z_j^*)(T_1^* - z_j^*)^{-1}(T_1 - z_\ell I)^{-1},\end{aligned} \tag{8.11}$$

and similarly for T_2. Eqs. (8.9) and (8.11) lead to

$$K_1^*(T_1^* - z_j^* I)^{-1}(T_1 - z_\ell)^{-1}K_1 = K_2^*(T_2^* - z_j^* I)^{-1}(T_2 - z_\ell)^{-1}K_2, \tag{8.12}$$

and hence to (8.10). Thanks to (8.12), we can define an operator U via

$$U(T_1 - z_\ell I)^{-1}K_1\phi = (T_2 - z_\ell I)^{-1}K_2\phi \quad (\phi \in E,\ \ell = 1, \ldots, m). \tag{8.13}$$

This operator is isometric and in fact unitary, thanks to the range conditions (8.7). Rewriting (8.9) as

$$K_1^*U^*U(T_1 - z_j I)^{-1}K_1\phi = K_2^*(T_2 - z_j I)^{-1}K_2\phi,$$

and taking into account the definition of U, we obtain that

$$K_1^*U^*U(T_1 - z_j I)^{-1}K_1\phi = K_2^*(T_2 - z_j I)^{-1}K_2\phi.$$

The range conditions (8.7) then lead to $K_1^*U^* = K_2^*$. We now show that $UT_1 = T_2U$. Condition (8.8) allows to write the power expansions

$$(T_1 - z_j I)^{-1} = -\frac{I}{z_j} - \frac{T_1}{z_j^2} - \frac{T_1^2}{z_j^3} - \cdots,$$

and

$$(T_2 - z_j I)^{-1} = -\frac{I}{z_j} - \frac{T_2}{z_j^2} - \frac{T_2^2}{z_j^3} - \cdots$$

for $j = 1, \ldots, n$. Definition (8.13) of U gives us

$$U\left(-\frac{I}{z_j} - \frac{T_1}{z_j^2} - \frac{T_1^2}{z_j^3} - \cdots\right)K_1 = \left(-\frac{I}{z_j} - \frac{T_2}{z_j^2} - \frac{T_2^2}{z_j^3} - \cdots\right)K_2,$$

and therefore

$$-\frac{UK_1}{z_j} + U\left(-\frac{T_1}{z_j^2} - \frac{T_1^2}{z_j^3} - \cdots\right)K_1 = -\frac{K_2}{z_j} + \left(-\frac{T_2}{z_j^2} - \frac{T_2^2}{z_j^3} - \cdots\right)K_2.$$

Since $K_2 = UK_1$ we obtain

$$U\frac{T_1}{z_j}\left(-\frac{I}{z_j} - \frac{T_1}{z_j^2} - \frac{T_1^2}{z_j^3} - \cdots\right)K_1 = \frac{T_2}{z_j}\left(-\frac{I}{z_j} - \frac{T_2}{z_j^2} - \frac{T_2^2}{z_j^3} - \cdots\right)K_2$$

so that

$$UT_1(T_1 - z_j I)^{-1}K_1 = T_2(T_2 - z_j I)^{-1}K_2, \quad j = 1, \ldots, n.$$

This last equation can be rewritten as

$$UT_1U^{-1}U(T_1 - z_j I)^{-1}K_1 = T_2(T_2 - z_j I)^{-1}K_2.$$

The range conditions (8.7) and the definition of U lead then to

$$UT_1U^{-1} = T_2,$$

which concludes the proof. □

Remark 8.5. In the previous theorem, condition (8.8) may be omitted when the channel operators K_1 and K_2 are invertible.

We also note that the theorem is still true for infinite dimensional Hilbert spaces $\mathscr{G}_1$ and $\mathscr{G}_2$ and for an infinite number of points.

Various properties of the operator T are reflected into properties of V_θ. This leads us to the next section and to the characterization of α-accretive operators.

9. Accretive and α-sectorial operators

The following families of operators were introduced by Kato [26, pp. 280, 318] and studied in details in [18].

Definition 9.1. Let $\alpha \in (0, \pi/2)$. A closed densely defined operator T acting on a Hilbert space $\mathscr{G}$ is called α-sectorial if for every ξ in the domain $\mathscr{D}$ of T, it holds that

$$(\cot\alpha)|\mathrm{Im}\langle T\xi,\xi\rangle_{\mathscr{G}}| \leqslant \mathrm{Re}\langle T\xi,\xi\rangle_{\mathscr{G}}. \tag{9.1}$$

The operator T is in particular accretive:

$$\mathrm{Re}\langle T\xi,\xi\rangle_{\mathscr{G}} \geqslant 0 \quad (\xi\in\mathscr{D}(T)). \tag{9.2}$$

The accretive operator is called *extremal* if it is not α sectorial for any $\alpha\in(0,\pi/2)$.

In the present work, we will focus on *bounded* α-sectorial operators and defer the treatment of the general case to a later publication. We characterize α-sectorial operators in terms of the positivity of associated kernels, which contain as special cases the kernels associated to Stieltjes functions. We show in Theorem 9.4 that if $N(z)$ is the Herglotz–Nevanlinna function associated to the *bounded* operator T (Definition 8.3), the operator is α-sectorial if and only if the following conditions are in force:

(1) The kernel (1.2) is positive in the sense of reproducing kernels in $\mathbb{C}_+$.

(2) The limit $\lim_{z\to\infty} N(z)$ is equal to 0.

(3) The measure $\mathrm{d}\mu$ in the representation (8.4) has compact support.

If the bounded operator T is α-sectorial, the exact value of the angle α can be computed via the formula

$$\cot\alpha = \inf_{\xi\in\mathscr{D}(T)} \frac{\mathrm{Re}\langle T\xi,\xi\rangle}{|\mathrm{Im}\langle T\xi,\xi\rangle|}. \tag{9.3}$$

When the state space is finite dimensional, the angle α can be given by another formula.

Theorem 9.2. *Let*

$$\theta = \begin{pmatrix} T & K & I \\ \mathscr{G} & & E \end{pmatrix}$$

be a Brodskii–Livsic time-invariant system with accretive main operator T (*with both $\mathscr{G}$ and E finite dimensional*) *and injective channel operator K. Then T is α-sectorial if and only if the limit*

$$\|V_\theta(0^-)\| \overset{\text{def.}}{=} \lim_{x\in\mathbb{R}^-,x\to 0} \|V_\theta(x)\|,$$

is finite. In this case, the angle α is given by the formula

$$\tan\alpha = \|V_\theta(0^-)\|. \tag{9.4}$$

Proof. Assume that the operator T is α-sectorial. By K^{-1} we mean the inverse of K on its range. It follows from (9.3) that

$$
\begin{aligned}
\cot\alpha &= \inf_{\xi\in\mathscr{D}(T)} \frac{\langle(\mathrm{Re}\ T)\xi,\xi\rangle}{\langle(\mathrm{Im}\ T)\xi,\xi\rangle}\\
&= \inf_{\xi\in\mathscr{D}(T)} \frac{\langle(\mathrm{Re}\ T)\xi,\xi\rangle}{\langle KK^*\xi,\xi\rangle}\\
&= \inf_{\xi\in\mathscr{D}(T)} \frac{\langle(\mathrm{Re}\ T)\xi,\xi\rangle}{\langle K^*\xi,K^*\xi\rangle}\\
&= \inf_{\xi\in\mathscr{D}(T)} \frac{\langle(\mathrm{Re}\ T)\ (K^{-1})^*K^*\xi,(K^{-1})^*K^*\xi\rangle}{\|K^*\xi\|^2}\\
&= \inf_{\xi\in\mathscr{D}(T)} \frac{\langle K^{-1}(\mathrm{Re}\ T)(K^{-1})^*K^*\xi,K^*\xi\rangle}{\|K^*\xi\|^2}\\
&= \lambda_{\min}\left(K^{-1}(\mathrm{Re}\ T)(K^{-1})^*\right)\\
&= \lim_{x\to 0^-}\lambda_{\min}\left(K^{-1}(\mathrm{Re}\ T-xI)(K^{-1})^*\right)\\
&= \lim_{x\to 0^-}\frac{1}{\lambda_{\max}\left(\left(K^{-1}(\mathrm{Re}\ T-xI)(K^{-1})^*\right)^{-1}\right)}\\
&= \lim_{x\to 0^-}\frac{1}{\lambda_{\max}\left(K^*(\mathrm{Re}\ T-xI)^{-1}K\right)}\\
&= \frac{1}{\|V_\theta(0^-)\|}. \qquad \square
\end{aligned}
$$

Corollary 9.3. *The operator T is extremal if and only if*

$$\lim_{x\to 0^-}\|V_\theta(x)\| = \infty.$$

The next theorem is of fundamental importance in the present work.

Theorem 9.4. *Let*

$$\theta = \begin{pmatrix} T & K & I\\ \mathscr{G} & & E\end{pmatrix}$$

be a Brodskii–Livsic time-invariant system with bounded main operator T. The operator T is α-sectorial if and only if for every set of nonreal points $z_1,\ldots,z_p$ (with $z_k \neq z_\ell{}^*$) and every set of vectors $h_1,\ldots,h_p\in E$, the following inequalities are valid:

$$\sum_{k,\ell=1}^{p}\left\langle \frac{V_\theta(z_k)-V_\theta(z_\ell)^*}{z_k-z_\ell{}^*}h_k,h_\ell\right\rangle_E \geqslant 0, \tag{9.5}$$

$$\sum_{k,\ell=1}^{p}\left\langle \frac{z_k V_\theta(z_k) - z_\ell{}^* V_\theta(z_\ell)^*}{z_k - z_\ell{}^*} h_k, h_\ell \right\rangle_E$$
$$\geqslant (\cot\alpha)\cdot \sum_{k,\ell=1}^{p} \langle V_\theta(z_\ell)^* \cdot V_\theta(z_k) h_k, h_\ell\rangle_E. \tag{9.6}$$

We first prove a preliminary lemma.

Lemma 9.5. *An element* $N \in \mathscr{S}^{\cot\alpha}$ *for which* $\lim_{z\to\infty} N(z) = 0$ *and* $\mathrm{d}\mu(t)$ *has compact support is of the form* (8.3).

Proof. The function N is in particular a Stieltjes function and admits the representation (see [27, Theorem A4, p. 392])

$$N(z) = Q + \int_0^\infty \frac{\mathrm{d}\mu(t)}{t-z}, \tag{9.7}$$

where Q is a nonnegative matrix and where $\mathrm{d}\mu$ is a positive matrix-valued measure on the positive line such that

$$\int_0^\infty \frac{\mathrm{d}\mu(t)}{1+t} < \infty.$$

Since $Q = \lim_{z\to\infty} N(z)$ the hypothesis in the lemma implies $Q = 0$ since $\mathrm{d}\mu$ is summable; just take for T the operator of multiplication by the independent variable in $\mathbf{L}_2^n(\mathrm{d}\mu)$ and K the imbedding of $\mathbb{C}^n$ into $\mathbf{L}_2^n(\mathrm{d}\mu)$. $\square$

Proof of Theorem 9.4. Writing $V_\theta(z) = K^*(\mathrm{Re}\,T - zI)^{-1}K$ we obtain

$$\begin{aligned}\frac{V_\theta(z_k) - V_\theta(z_\ell)^*}{z_k - z_\ell^*} &= \frac{1}{z_k - z_\ell^*}\Big(K^*(\mathrm{Re}\,T - z_k I)^{-1}K - K^*(\mathrm{Re}\,T - z_\ell^* I)^{-1}K\Big)\\ &= K^*((\mathrm{Re}\,T - z_\ell I)^{-1})^*(\mathrm{Re}\,T - z_k I)^{-1}K.\end{aligned}$$

Set now

$$\xi_k = (\mathrm{Re}\,T - z_k I)^{-1} K h_k \quad (k = 1, 2, \ldots, n), \tag{9.8}$$

and

$$\xi = \sum_{1}^{n} \xi_k. \tag{9.9}$$

We get

$$\sum_{k,\ell=1}^{n}\left\langle \frac{V_\theta(z_k) - V_\theta(z_\ell)^*}{z_k - z_\ell^*} h_k, h_\ell \right\rangle_E$$
$$= \sum_{k,\ell=1}^{n} \langle K^*(\mathrm{Re}\,T - z_\ell^* I)^{-1}(\mathrm{Re}\,T - z_k I)^{-1} K h_k, h_\ell\rangle_E$$

$$= \sum_{k,\ell=1}^{n} \langle \xi_k, \xi_\ell \rangle_{\mathscr{G}}$$
$$= \langle \xi, \xi \rangle_{\mathscr{G}} \geqslant 0.$$

Since the operator T is α-sectorial, then

$$\cot\alpha \cdot |\langle \mathrm{Im}\ T\xi, \xi\rangle| \leqslant \langle \mathrm{Re}\ T\xi, \xi\rangle. \tag{9.10}$$

From (9.8) and (9.9) follows

$$\sum_{k,\ell=1}^{n} \langle \mathrm{Re}\ T\xi_k, \xi_\ell\rangle \geqslant \cot\alpha \cdot \sum_{k,\ell} \langle \mathrm{Im}\ T\xi_k, \xi_\ell\rangle. \tag{9.11}$$

Since $\mathrm{Im}\ T = KK^*$, we get

$$\begin{aligned} &\sum_{k,\ell=1} \langle \mathrm{Re}\ T(\mathrm{Re}\ T - z_k I)^{-1} K h_k, (\mathrm{Re}\ T - z_\ell I)^{-1} K h_\ell\rangle \\ &\quad \geqslant \cot\alpha \cdot \sum_{k,\ell} \langle KK^*(\mathrm{Re}\ T - z_k I)^{-1} K h_k, (\mathrm{Re}\ T - z_\ell I)^{-1} K h_\ell\rangle, \end{aligned} \tag{9.12}$$

which leads to

$$\begin{aligned} &\sum_{k,\ell=1} \langle K^*(\mathrm{Re}\ T - z_\ell^* I)^{-1} \mathrm{Re}\ T(\mathrm{Re}\ T - z_k I)^{-1} K h_k, h_\ell\rangle_E \\ &\quad \geqslant \cot\alpha \cdot \sum_{k,\ell=1} \langle K^*(\mathrm{Re}\ T - z_\ell^* I)^{-1} KK^*(\mathrm{Re}\ T - z_k I)^{-1} K h_k, h_\ell\rangle_E. \end{aligned}$$

Since

$$\begin{aligned} &K^*(\mathrm{Re}\ T - z_\ell^* I)^{-1} \mathrm{Re}\ T(\mathrm{Re}\ T - z_k I)^{-1} K \\ &\quad = K^* \frac{z_\ell^*(\mathrm{Re}\ T - z_\ell^* I)^{-1} - z_k K^*(\mathrm{Re}\ T - z_k I)^{-1}}{z_\ell^* - z_k} K, \end{aligned}$$

we obtain (9.5). We now turn to the converse statement, i.e., starting from a function for which (9.5) and (9.6) hold, and showing that it is the Herglotz–Nevanlinna operator associated to a α-sectorial operator. By Lemma 9.5, the function N is of the form (9.7) with $Q = 0$, i.e.,

$$N(z) = \int_0^\infty \frac{\mathrm{d}\mu(t)}{t - z},$$

which we rewrite as $N(z) = K^*(A - zI)^{-1}K$ with A self-adjoint. Set $T = A + \mathrm{i}KK^*$. The computations of the first part of the proof of the theorem can be read backwards to lead that T is α-sectorial. □

We remark that, when $\alpha = \pi/2$, inequalities (9.6) and (9.5) express that V_θ is a Stieltjes function.

10. Interpolation and Brodskii–Livsic scattering systems of minimal dimension

In this section, we gather applications of the interpolation results proved above to the theory of Brodskii–Livsic scattering systems. The main idea is as follows: *solve an interpolation problem in the class* $\mathscr{S}^{\cot\alpha}$ *and look for values of the parameter for which the solutions are rational and tend to* 0 *at infinity*. *Then, we have Brodskii–Livsic scattering systems*.

The following result will be used in the sequel:

Proposition 10.1. *Let J be given by* (1.7) *and let W be a rational J-inner function, i.e.*,

$$W(z)^* J W(z) \leqslant J, \quad z \in \mathbb{C}_+, \tag{10.1}$$

$$W(z)^* J W(z) = J, \quad z \in \mathbb{R} \tag{10.2}$$

(*z being a point of analyticity of W*). *Let* $W = (W_{ij})_{i,j=1,2}$ *be the decomposition of W into* $\mathbb{C}^{n\times n}$*-valued blocks. Let* N_0 *be a constant self-adjoint matrix and assume*

$$\det(W_{21}(z)N_0 + W_{22}(z)) \not\equiv 0. \tag{10.3}$$

Then, the matrix-valued function

$$N(z) = (W_{11}(z)N_0 + W_{12}(z))(W_{21}(z)N_0 + W_{22}(z))^{-1} \tag{10.4}$$

is a Herglotz–Nevanlinna function of McMillan degree $\deg W$.

Before giving the proof, we recall the notion of McMillan degree. A rational matrix-valued function $r(z)$ analytic at infinity can be written in the form

$$r(z) = D + C(zI_m - A)^{-1}B,$$

where A, B, C and D are matrices of appropriate dimensions. The smallest dimension for the matrix A in this representation is called the McMillan degree of r. We refer to [11] for more information.

Proof. To check that N defined by (10.4), it suffices to remark that

$$(N_0\ I_n)W(z)^* J W(z) \begin{pmatrix} N_0 \\ I_n \end{pmatrix} \leqslant (N_0\ I_n) J \begin{pmatrix} N_0 \\ I_n \end{pmatrix}, \quad z \in \mathbb{C}_+, \tag{10.5}$$

$$(N_0\ I_n)W(z)^* J W(z) \begin{pmatrix} N_0 \\ I_n \end{pmatrix} = (N_0\ I_n) J \begin{pmatrix} N_0 \\ I_n \end{pmatrix}, \quad z \in \mathbb{R}, \tag{10.6}$$

and that

$$(N_0\ I_n) J \begin{pmatrix} N_0 \\ I_n \end{pmatrix} = 0.$$

The claim on the degree uses reproducing kernel arguments as in [7,8]. The reproducing kernel Hilbert space $\mathscr{H}(W)$ with reproducing kernel $(J - W(w)^* J W(z)/$

$-\mathrm{i}(z-w^*))$ has dimension deg$W$, and the map $F \mapsto F\binom{N(z)}{I_n}$ is unitary from $\mathscr{H}(W)$ onto the reproducing kernel Hilbert space with reproducing kernel $(N(z)-N(w)^*/z-w^*)$. Finally, the dimension of this last space is equal to the McMillan degree of N. $\square$

In the following theorem, the interpolation values are complex numbers.

Theorem 10.2. *Let $z_1,\ldots,z_p \in \mathbb{C}_+$ and $v_1,\ldots,v_p$ be interpolation data and assume that the matrix with ℓ, j entry equal to $(v_\ell - v_j^*)/(z_\ell - z_j^*)$ is strictly positive. Then there exists a prime interpolation Brodskii–Livsic system*

$$\theta = \begin{pmatrix} T & K & J \\ \mathscr{G} & & \mathbb{C} \end{pmatrix},$$

with $\dim \mathscr{G} = p$.

The main operator T is α-sectorial if and only if the $p \times p$ hermitian matrix with ℓ, j entry equal to

$$\frac{z_\ell v_\ell - z_j^* v_j^*}{z_\ell - z_j^*} - \cot\alpha \; v_\ell v_j^*$$

is nonnegative. Assume this matrix strictly positive. Then, there is a solution with state space of dimension p.

Proof. The existence of a function N in the class $\mathscr{S}^{\cot\alpha}$ for which $N(z_j) = v_j$ for $j = 1,\ldots,p$ follows from the general theorems stated above. From Corollary 5.6 we see that we can take the choice of parameter $G \equiv 0$ to obtain the required function:

$$N(z) = W_{12}(z)W_{22}(z)^{-1}. \tag{10.7}$$

This function has McMillan degree p, and so $\dim \mathscr{G} = p$. $\square$

More generally:

Theorem 10.3. *In the notation and with the hypothesis in Theorem* 5.5, *the interpolation problem has a solution of McMillan degree, the degree of W, with $N(\infty) = 0$, and so N is the Herglotz–Nevanlinna function associated to a finite dimensional α-sectorial operators.*

It suffices to take $G \equiv 0$.

11. Interpolation and Brodskii–Livsic scattering: explicit solutions

The results in the present section are mostly consequences of the main results of the first part of the paper. They deal with the scalar case and we include them for illustration of the general theory. Also, we provide explicit constructions of the various systems. In this and the next sections, we denote by $\{z_k,\ k=1,\dots,m\}$ and $\{v_k,\ k=1,\dots,m\}$ an interpolation data (with $z_k \neq z_j^*$) and we set

$$\mathbb{P} = \left(\frac{v_k - v_j^*}{z_k - z_j^*}\right)_{k,j=1,\dots,m}, \quad \mathbb{Q} = \left(\frac{z_k v_k - z_j^* v_j^*}{z_k - z_j^*}\right)_{k,j=1,\dots,m}. \tag{11.1}$$

Theorem 11.1. *Let $\{z_k,\ k=1,\dots,m\} \in \mathbb{C}_+$ and $\{v_k,\ k=1,\dots,m\}$ be an interpolation data for which the matrix $\mathbb{P}$ is strictly positive. Then there exists an interpolation Brodskii–Livsic scattering system θ with state space of dimension m solution of the corresponding interpolation problem. The main operator is α-sectorial if and only if*

$$\left(\frac{z_k v_k - z_j^* v_j^*}{z_k - z_j^*}\right)_{k,j=1,\dots,m} \geqslant \cot\alpha \left(v_k v_j^*\right)_{k,j=1,\dots,m}. \tag{11.2}$$

Proof. Consider the space $\mathbb{C}^m$ with the inner product

$$\langle \xi, \eta\rangle_{\mathbb{C}^m_{\mathbb{P}}} = \eta^* \mathbb{P} \xi. \tag{11.3}$$

It is readily seen that the operator $\vec{A} = \mathbb{P}^{-1}\mathbb{Q}$ is self-adjoint with respect to this inner product. Set

$$\vec{g} = \mathbb{P}^{-1}\varphi, \quad \varphi = \begin{pmatrix} v_1^* \\ \vdots \\ v_m^* \end{pmatrix}. \tag{11.4}$$

The operators

$$\vec{T} = \vec{A} + \mathrm{i}\langle\cdot, \vec{g}\rangle_{\mathbb{C}^m_{\mathbb{P}}} \vec{g}, \tag{11.5}$$

$$\vec{K} c = c \cdot \vec{g} \tag{11.6}$$

satisfy Im $\vec{T} = \vec{K}\vec{K}^*$ and therefore define a Brodskii–Livsic scattering system $\vec{\theta}$. Set

$$\begin{aligned} V_{\vec{\theta}}(z) &= \vec{K}^* (\vec{A} - zI)^{-1} \vec{K} \\ &= \vec{K}^* (\mathrm{Re}\ \vec{T} - zI)^{-1} \vec{K}. \end{aligned} \tag{11.7}$$

We will denote by the same notation the representation of this operator by a scalar.

We check that $V_{\vec{\theta}}(z_k) = v_k,\ k=1,\dots,m$. We have that

$$(\mathbb{Q} - z_k \mathbb{P})_{j,k} = \frac{z_k v_k - z_j^* v_j^*}{z_k - z_j^*} - z_k \frac{v_k - v_j^*}{z_k - z_j^*} = v_j^*. \tag{11.8}$$

From (11.8), it follows that the kth column of the matrix $\mathbb{P} - z_k\mathbb{Q}$ is equal to φ (defined in (11.4)). Hence, $(\overrightarrow{A} - z_k I_m)e_k = \overrightarrow{g}$ (where e_k denotes the $m \times 1$ column vector whose all entries are equal to 0, besides the kth one, which is equal to 1). Therefore,

$$\begin{aligned} V_{\overrightarrow{\theta}}(z_k) &= \left\langle \left(\overrightarrow{A} - z_k I_m\right)^{-1} \overrightarrow{g}, \overrightarrow{g} \right\rangle_{\mathbb{C}^m_{\mathbb{P}}} \\ &= \langle e_k, g\rangle_{\mathbb{C}^m_{\mathbb{P}}} \\ &= \left\langle \mathbb{P}e_k, \mathbb{P}^{-1}\overrightarrow{g} \right\rangle_{\mathbb{C}^m_{\mathbb{P}}} \\ &= \left\langle \mathbb{P}e_k, \mathbb{P}^{-1}\overrightarrow{g} \right\rangle_{\mathbb{C}^m} \\ &= \left\langle e_k, \overrightarrow{g} \right\rangle_{\mathbb{C}^m} \\ &= v_k. \end{aligned}$$

So we proved that the Brodskii–Livsic scattering system of the form (11.5) and (11.6) is the interpolation system for the Nevanlinna–Pick interpolation problem. If the operator $\overrightarrow{T}$ is α-sectorial, it follows from Theorem 9.4 that (11.2) holds.

Suppose conversely that (11.2) holds. We prove that the operator $\overrightarrow{T}$ is α-sectorial. Condition (11.2) implies that for an arbitrary vector $\xi = (\xi_j) \in \mathbb{C}^m$, we have the inequality

$$\cot\alpha \sum_{k,j=1}^{m} v_k v_j^* \xi_k \xi_j^* \leqslant \sum_{k,j=1}^{m} \frac{z_k v_k - z_j^* v_j^*}{z_k - z_j^*} \xi_k \xi_j^*,$$

i.e.,

$$\cot\alpha \left(\sum_1^n v_k \xi_k\right)\left(\sum_1^n v_j \xi_j\right)^* \leqslant \langle \mathbb{Q}\xi, \xi\rangle_{\mathbb{C}^n}.$$

This inequality can be rewritten as

$$\cot\alpha \left(\sum_1^n v_k \xi_k\right)\left(\sum_1^n v_j \xi_j\right)^* \leqslant \langle \overrightarrow{A}\xi, \xi\rangle_{\mathbb{C}^m_{\mathbb{P}}}. \tag{11.9}$$

Besides, we have

$$\langle \xi, \overrightarrow{g}\rangle_{\mathbb{C}^m_{\mathbb{P}}} = \langle \mathbb{P}\xi, \mathbb{P}^{-1}\varphi\rangle_{\mathbb{C}^n} = \langle \xi, \varphi\rangle_{\mathbb{C}^m}. \tag{11.10}$$

It follows from (11.9) and (11.10) that

$$\cot\alpha |\langle \xi, \overrightarrow{g}\rangle_{\mathbb{C}^m_{\mathbb{P}}}|^2 \leqslant \langle \overrightarrow{A}\xi, \xi\rangle_{\mathbb{C}^m_{\mathbb{P}}},$$

or

$$\cot\alpha |\mathrm{Im}\langle \overrightarrow{T}\xi, \xi\rangle_{\mathbb{C}^m_{\mathbb{P}}}| \leqslant \mathrm{Re}\langle \overrightarrow{T}\xi, \xi\rangle_{\mathbb{C}^m_{\mathbb{P}}}.$$

Thus the operator $\overrightarrow{T}$ is α-sectorial. □

Theorem 11.2. *Let $z_1, \ldots, z_m \in \mathbb{C}_+$ and $v_1, \ldots, v_m$ be an interpolation data. If the Brodskii–Livsic scattering system*

$$\theta = \begin{pmatrix} T & K & I \\ \mathscr{G} & & \mathbb{C} \end{pmatrix}, \quad \operatorname{Im} T = KK^*,$$

is an interpolation system for the Nevanlinna–Pick interpolation problem with dim $\mathscr{G} = m$, *if the z_k are not eigenvalues of the main operator T and if it holds that*

$$\bigvee_{k=1}^{m} (T - z_k I)^{-1} K\mathbb{C} = \mathbb{C}^m, \tag{11.11}$$

then the matrix $\mathbb{P}$ is strictly positive. The given system is unitarily equivalent to the system $\vec{\theta}$ of the form (11.5) *and* (11.6) *and*

$$\bigvee_{k=1}^{m} \operatorname{Ran}(\vec{T} - z_k I)^{-1} \vec{K} = \mathbb{C}^n_{\mathbb{P}}. \tag{11.12}$$

Proof. Set $V_\theta(z) = K^*(\operatorname{Re} T - zI)^{-1}K$. Then,

$$\begin{aligned}
\frac{V_\theta(z_k) - V_\theta(z_j)^*}{z_k - z_j^*} &= K^*(\operatorname{Re} T - z_j^* I)^{-1}(\operatorname{Re} T - z_k I)^{-1} K \\
&= K^*(\operatorname{Re} T - z_j^* I)^{-1} K K^{-1}(K^{-1})^* K^*(\operatorname{Re} T - z_k I)^{-1} K \\
&= x^2 V_\theta(z_j)^* V_\theta(z_k),
\end{aligned}$$

where we have set $K^{-1}(K^{-1})^* = x^2 I$ (K^{-1} denotes the inverse of K on its range). Since the function $V_\theta(z)$ solves the interpolation problem associated to the interpolation data we have

$$\frac{v_k - v_j^*}{z_k - z_j^*} = x^2 v_j^* v_k. \tag{11.13}$$

We now show that the matrix $\mathbb{P}$ is nonsingular. Let $\xi = (\xi_j) \in \mathbb{C}^m$ be in the kernel of $\mathbb{P}$. Then,

$$\sum_{k=1}^{m} \frac{v_k - v_j^*}{z_k - z_j^*} \xi_k = 0, \quad j = 1, \ldots, m. \tag{11.14}$$

Equation (11.13) then implies that

$$\sum_{k=1}^{m} x^2 v_j^* v_k \xi_k = 0, \quad j = 1, \ldots, m,$$

and hence

$$\sum_{k=1}^{m} v_k \xi_k = 0, \tag{11.15}$$

i.e., $\sum_{k=1}^{m} K^*(\mathrm{Re}\ T - z_k I)^{-1} K\xi_k = 0$. Thus, we have

$$\sum_{k=1}^{m} (\mathrm{Re}\ T - z_k I)^{-1} K\xi_k = 0. \tag{11.16}$$

The operator T has a one-dimensional imaginary part. Therefore T can be represented in the form

$$T = \mathrm{Re}\ T + \mathrm{i}\langle \cdot, g\rangle g. \tag{11.17}$$

It follows from (11.11) that the vectors

$$x_k = 2(T - z_k I)^{-1} g, \quad k = 1, \ldots, m, \tag{11.18}$$

are linearly independent. It follows from (11.17) that

$$x_k = (2 - \mathrm{i}\langle x_k, g\rangle)(\mathrm{Re}\ T - z_k I)^{-1} g. \tag{11.19}$$

Set

$$\omega(z) = 1 - 2\mathrm{i}\langle (T - zI)^{-1} g, g\rangle. \tag{11.20}$$

We then have

$$x_k = (1 + \omega(z_k))(\mathrm{Re}\ T - z_k I)^{-1} g. \tag{11.21}$$

The function ω is the characteristic operator-valued function of T; since the z_k are not the eigenvalues of T, we have $|\omega(z_k)| > 1$. We may suppose $K(1) = g$ and therefore (11.16) implies that

$$\sum_{k=1}^{m} \frac{\xi_k}{1 + \omega(z_k)} x_k = 0.$$

Since the vectors x_k are linearly independent, we obtain that $\xi_k/(1 + \omega(z_k)) = 0$, and therefore $\xi_k = 0$, that is, $\mathbb{P}$ is nonsingular. Therefore we can consider the Brodskii–Livsic scattering system $\vec{\theta}$ of the form (11.5) and (11.6). Since $\vec{\theta}$ is also an interpolation system we have that

$$V_{\vec{\theta}}(z_k) = v_k. \tag{11.22}$$

Thus

$$V_{\vec{\theta}}(z_k) = V_{\theta}(z_k) = v_k, \quad k = 1, \ldots, m, \tag{11.23}$$

and

$$K^*(\mathrm{Re}\ T - z_k I)^{-1} K = \vec{K}^*(\mathrm{Re}\ \vec{T} - z_k I)^{-1} \vec{K}, \quad k = 1, \ldots, m. \tag{11.24}$$

Hence we obtain that

$$W_{\theta}(z_k) = W_{\vec{\theta}}(z_k), \quad k = 1, \ldots, m, \tag{11.25}$$

that is

$$K^*(T - z_k I)^{-1}K = \vec{K}^*(\vec{T} - z_k I)^{-1}\vec{K}, \quad k = 1, \ldots, m. \tag{11.26}$$

The operator U defined by

$$U(T - z_k I)^{-1}K = (\vec{T} - z_k I)^{-1}\vec{K} \tag{11.27}$$

is isometric from $\mathscr{G}$ onto $\mathbb{C}_{\mathbb{P}^m}$ and satisfies

$$\vec{K} = UK. \tag{11.28}$$

We leave to the reader to check that (11.27) implies that

$$UT = \vec{T}U \tag{11.29}$$

and therefore θ and $\vec{\theta}$ are unitarily equivalent. Hence $V_\theta(z) = V_{\vec{\theta}}(z)$ and the theorem is proved. □

Theorem 11.3. *Let z_k and v_k be an interpolation data with $z_k \in \mathbb{C}_+$ for which the matrix $\mathbb{P} = ((v_k - v_j^*)/(z_k - z_j^*))$ is strictly positive. Assume that z_{k_i}, $i = 1, \ldots, p$, are eigenvalues of T. Then,*

$$v_{k_1} = \cdots = v_{k_p} = \mathrm{i}. \tag{11.30}$$

Proof. If $\mathbb{P}$ is invertible, Theorem 11.1 implies that there exists a Brodskii–Livsic scattering system $\vec{\theta}$ of the form (11.5) and (11.6) which is an interpolation system for the given system. Thus,

$$V_{\vec{\theta}}(z_k) = v_k, \quad k = 1, \ldots, n,$$

and

$$V_{\vec{\theta}}(z_k) = \vec{K}^*(\mathrm{Re}\ \vec{T} - z_k I)^{-1}\vec{K}.$$

Recall that

$$W_{\vec{\theta}}(z) = \mathrm{i}\frac{V_{\vec{\theta}}(z) - 1}{V_{\vec{\theta}}(z) + 1}. \tag{11.31}$$

It is proved in the theory of nonselfadjoint operators that the spectrum of the operator $\vec{T}$ coincides with the set of singular points of $W_{\vec{\theta}}$. Therefore the z_{k_i} are poles of $W_{\vec{\theta}}$ and $\lim_{z \to z_{k_m}} W_{\vec{\theta}} = \infty$. Hence

$$v_{k_m} = \lim_{z \to z_{k_m}} V_{\vec{\theta}}(z) = \mathrm{i},$$

thanks to (11.31). □

The converse of this theorem is now considered.

Theorem 11.4. *Let $z_1, \ldots, z_m$ and $v_1, \ldots, v_m$ be interpolation data with $z_k \in \mathbb{C}_+$. Assume the matrix $\mathbb{P} = ((v_k - v_j^*)/(z_k - z_j^*))$ is strictly positive. Assume that* (11.30) *holds. Then* $z_{k_1}, \ldots, z_{k_p}$ *are eigenvalues of the operator T defined by* (11.5) *and* (11.6).

Proof. If the matrix $\mathbb{P}$ is strictly positive, Theorem 11.1 implies that the interpolation Nevanlinna–Pick problem is solvable with a Brodskii–Livsic scattering system of the form (11.5) and (11.6). Thus,

$$V_{\vec{\theta}}(z_{k_m}) = i, \quad m = 1, \ldots, p.$$

Since

$$W_{\vec{\theta}}(z) = \frac{1 - \mathrm{i}V_{\vec{\theta}}(z)}{1 + \mathrm{i}V_{\vec{\theta}\,(z)}}, \tag{11.32}$$

we get

$$\lim_{z \to z_{k_j}} W_{\vec{\theta}}(z) = \lim_{z \to z_{k_j}} \frac{1 - \mathrm{i}V_{\vec{\theta}}(z)}{1 + \mathrm{i}V_{\vec{\theta}}(z)} = \infty, \quad m = 1, \ldots, p.$$

This means that the z_{k_j} are poles of $W_{\vec{\theta}}(z)$, i.e., of the characteristic function of $\vec{T}$. Therefore, they are eigenvalues of $\vec{T}$. $\square$

Theorem 11.5. *Let* $z_k,\ k = 1, \ldots, m$ *and* $v_k = i,\ k = 1, \ldots, m$, *be an interpolation data with* $z_k \in \mathbb{C}_+$. *Then there exists a unique* (*up to unitary equivalence*) *Brodskii–Livsic scattering system solution of the corresponding interpolation problem with state space of dimension m.*

Proof. Set $W(z) = \prod_{k=1}^{m} (z - z_k^*)/(z - z_k)$ and $T_\triangle$ to be

$$T_\triangle = \begin{pmatrix} z_1 & \mathrm{i}\sqrt{2\ \mathrm{Im}\ z_1}\sqrt{2\ \mathrm{Im}\ z_2} & \cdots & \mathrm{i}\sqrt{2\ \mathrm{Im}\ z_1}\sqrt{2\ \mathrm{Im}\ z_m} \\ 0 & z_2 & \cdots & \mathrm{i}\sqrt{2\ \mathrm{Im}\ z_2}\sqrt{2\ \mathrm{Im}\ z_m} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & z_m \end{pmatrix}. \tag{11.33}$$

Then,

$$\mathrm{Im}\ T_\triangle = g_\triangle^* \cdot g_\triangle \tag{11.34}$$

with

$$g_\triangle = \begin{pmatrix} \sqrt{\mathrm{Im}\ z_1} \\ \sqrt{\mathrm{Im}\ z_2} \\ \vdots \\ \sqrt{\mathrm{Im}\ z_m} \end{pmatrix}. \tag{11.35}$$

Consider the Brodskii–Livsic system

$$\theta_\triangle = \begin{pmatrix} T_\triangle & K_\triangle & 1 \\ \mathbb{C}^m & & \mathbb{C} \end{pmatrix}, \tag{11.36}$$

where $K_\triangle(c) = c \cdot g_\triangle$, $c \in \mathbb{C}$. The characteristic function of the triangular system $\theta_\triangle$ is

$$W_{\Theta_\triangle}(z) = 1 - 2\mathrm{i}g_\triangle^*(T_\triangle - zI)^{-1}g_\triangle = \prod_{k=1}^{m} \frac{z - z_k^*}{z - z_k}. \tag{11.37}$$

The function

$$V_{\Theta_\triangle}(z) = \mathrm{i}\frac{W_{\Theta_\triangle}(z) - 1}{W_{\Theta_\triangle}(z) + 1} \tag{11.38}$$

clearly satisfies the interpolation conditions

$$V_{\Theta_\triangle}(z_k) = i, \quad k = 1, \ldots, m. \tag{11.39}$$

Therefore the triangular Brodskii–Livsic scattering system (11.36) is the interpolation system for the stated Nevanlinna–Pick interpolation problem.

Let the system

$$\theta = \begin{pmatrix} T & K & J \\ \mathscr{G} & & \mathbb{C} \end{pmatrix}, \quad \dim \mathscr{G} = m, \quad \operatorname{Im} T = KK^*,$$

be the interpolation system (Brodskii–Livsic scattering system) in the Nevanlinna–Pick interpolation problem for the data $z_k, k = 1, \ldots, n$ and $v_k = i, k = 1, \ldots, m$. Without loss of generality, we will assume that $\operatorname{Im} T = (\cdot\,,\ \cdot)g$ and $K(1) = g$. Then, $K^*(x) = (x, g)_{\mathscr{G}}g$. Define $V_\theta(z)$ and $W_\theta(z)$ as above, with this choice of T and K. Obviously (since θ is the interpolation system),

$$V_\theta(z_k) = \mathrm{i}, \quad k = 1, \ldots, m, \tag{11.40}$$

and

$$\lim_{z \to z_{k_j}} \frac{1 - \mathrm{i}V_\theta(z)}{1 + \mathrm{i}V_\theta(z)} = \infty, \quad j = 1, \ldots, m.$$

Therefore, each point z_k is a pole of $W_\theta(z)$. According to the well-known result of Livsic [17,28], the z_k are exactly the points of the spectrum of T since $\dim \mathscr{G} = n$. Therefore, by the same result of Livsic,

$$W_\theta(z) = \prod_{1}^{m} \frac{z - z_k^*}{z - z_k} = W_{\theta_\triangle}(z). \tag{11.41}$$

By the theorem of Livsic about unitary equivalence of systems, the systems θ and $\theta_\triangle$ are unitary equivalent and therefore

$$V_\theta(z) = V_{\theta_\triangle}(z). \qquad \square \tag{11.42}$$

Corollary 11.6. *Let z_k be arbitrary points in the open upper half-plane. There exists a unique matrix U such that*

$$U^*\mathbb{P}U = I_m,$$

$$U\left(\frac{2\mathrm{i}}{z_k - z_j^*}\right)^{-1}\left(\mathrm{i}\left(\frac{z_k + z_j^*}{z_k - z_j^*}\right) - \mathrm{i}\Pi\right) = T_\triangle U,$$

and

$$U\left(\frac{2\mathrm{i}}{z_k - z_j^*}\right)^{-1}\begin{pmatrix}-\mathrm{i}\\ -\mathrm{i}\\ \vdots\\ -\mathrm{i}\end{pmatrix} = \begin{pmatrix}\sqrt{\mathrm{Im}\, z_1}\\ \vdots\\ \sqrt{\mathrm{Im}\, z_m}\end{pmatrix}, \tag{11.43}$$

where

$$\Pi = \begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \vdots & & & \vdots\\ 1 & 1 & \cdots & 1\end{pmatrix},$$

and $T_\triangle$ is of the form (11.33).

Proof. The matrix $\mathbb{P} = (2\mathrm{i}/(z_k - z_j^*))$ is strictly positive and the function $V_{\theta_\triangle}(z)$ defined by (11.38) satisfies $V_{\theta_\triangle}(z_k) = \mathrm{i}$ for $k = 1, \ldots, n$. It has been shown that the system $\overrightarrow{\theta}$ of the form (11.5) and (11.6) also generates a solution of this Nevanlinna–Pick interpolation problem, of the form

$$V_{\overrightarrow{\theta}}(z) = \langle(\overrightarrow{A} - zI)^{-1}\overrightarrow{g}, \overrightarrow{g}\rangle_{\mathbb{C}^n_{\mathbb{P}}},$$

where

$$\begin{aligned}\overrightarrow{A} &= \mathbb{P}^{-1}\left(\frac{z_k + z_j^*}{z_k - z_j^*}\right),\\ \overrightarrow{g} &= \mathbb{P}^{-1}\begin{pmatrix}-\mathrm{i}\\ \vdots\\ -\mathrm{i}\end{pmatrix}.\end{aligned} \tag{11.44}$$

If we denote once again

$$V_{\overrightarrow{\theta}}(z) = \overrightarrow{K}^*(\mathrm{Re}\,\overrightarrow{T} - zI)^{-1}\overrightarrow{K}, \tag{11.45}$$

we get that $V_{\overrightarrow{\theta}}(z_k) = \mathrm{i}$ and

$$W_{\theta_\triangle}(z) = W_{\overrightarrow{\theta}}(z), \tag{11.46}$$

where $W_{\theta_\triangle}(z)$ has the form (11.37) and $W_{\overrightarrow{\theta}}(z)$ has the form

$$W_{\overrightarrow{\theta}}(z) = I - 2\mathrm{i}\,\overrightarrow{K}^*(\overrightarrow{T} - zI)^{-1}\overrightarrow{K}. \tag{11.47}$$

There exits thus a unique unitary mapping U from $\mathbb{C}^n_{\mathbb{B}}$ onto $\mathbb{C}^n$ such that

$$U\overrightarrow{T} = T_{\triangle}U, \quad U\overrightarrow{K} = K_{\triangle}. \tag{11.48}$$

The fact that U is an isometry means that

$$U^*\mathbb{P}U = I. \tag{11.49}$$

Taking into account (11.33), (11.36), (11.5) and (11.6) we get that the relations (11.48) and (11.49) coincide with (11.43). □

Theorem 11.7. *Let z_k and v_k be an interpolation data with $z_k \in \mathbb{C}_+$. Assume the matrix $\mathbb{P}$ strictly positive and the matrix $\mathbb{Q}$ nonnegative. The operator $\overrightarrow{T}$ defined by* (11.5) *is extremal if and only if* $\det \mathbb{Q} = 0$ *and for the unitary operator U on $\mathbb{C}^n_{\mathbb{P}}$ for which*

$$U\overrightarrow{A}U^* = U\mathbb{P}^{-1}\mathbb{Q}U^* = \mathrm{diag}(0, \ldots 0, \lambda_1, \ldots, \lambda_q) \tag{11.50}$$

(*with $\lambda_i > 0$, p elements equal to* 0 *and $p + q = n$*) *at least one from the first p coordinates of the vector $U\overrightarrow{g}$ with respect to the orthonormal basis of eigenvectors of the self-adjoint operator $\overrightarrow{A} = \mathbb{P}^{-1}\mathbb{Q}$ is not equal to* 0.

Proof. Set

$$V_\theta(z) = \langle(\mathbb{P}^{-1}\mathbb{Q} - zI)^{-1}\overrightarrow{g}, \overrightarrow{g}\rangle_{\mathbb{C}^m_{\mathbb{P}}}, \tag{11.51}$$

and let $\overrightarrow{T}$ be of the form (11.5) be an extremal operator. From Theorem 9.2 we have that

$$\lim_{x\to 0^-} \|V_\theta(x)\| = \lim_{x\to 0^-} \langle(\mathbb{P}^{-1}\mathbb{Q} + xI)^{-1}\overrightarrow{g}, \overrightarrow{g}\rangle_{\mathbb{C}^m_{\mathbb{P}}} = \infty. \tag{11.52}$$

Assume that $\mathbb{Q}$ is invertible. Then,

$$\lim_{x\to 0^-} (\mathbb{P}^{-1}\mathbb{Q} + xI)^{-1} = \mathbb{Q}^{-1}\mathbb{P}, \tag{11.53}$$

and the limit (11.52) would then finite. Therefore, $\det \mathbb{Q} = 0$. Since the operator $\overrightarrow{A}$ is self-adjoint, there exists a unitary operator U on $\mathbb{C}^n_{\mathbb{P}}$ for which (11.50) holds. Suppose that all the p first coordinates of the vector $U\overrightarrow{g}$ with respect to this orthonormal basis are equal to 0, i.e.,

$$U\overrightarrow{g} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_1 \\ \vdots \\ \xi_q \end{pmatrix},$$

with $\xi_i \neq 0$. We have

$$\begin{aligned}
\|V_{\vec{\theta}}(0^-)\| &= \lim_{\varepsilon\to 0}\langle(\mathbb{P}^{-1}\mathbb{Q}+\varepsilon I)^{-1}\vec{g}\,,\ \vec{g}\,\rangle_{\mathbb{C}^n_{\mathbb{P}}}\\
&= \lim_{\varepsilon\to 0}\langle U(\mathbb{P}^{-1}\mathbb{Q}+\varepsilon I)^{-1}U^*U\vec{g}\,, U\vec{g}\,\rangle_{\mathbb{C}^n_{\mathbb{P}}}\\
&= \lim_{\varepsilon\to 0}\langle(U(\mathbb{P}^{-1}\mathbb{Q}+\varepsilon I)U^*)^{-1}U\vec{g}\,, U\vec{g}\,\rangle_{\mathbb{C}^n_{\mathbb{P}}}\\
&= \lim_{\varepsilon\to 0}\left\langle\begin{pmatrix}\frac{1}{\varepsilon} & 0 & 0 & & \\ & \ddots & & & \\ 0 & \frac{1}{\varepsilon} & 0 & 0 & \\ & & & \frac{1}{\lambda_1+\varepsilon} & \\ & & & & \frac{1}{\lambda_q+\varepsilon}\end{pmatrix}\begin{pmatrix}0\\ \vdots\\ 0\\ \xi_1\\ \vdots\\ \xi_q\end{pmatrix},\begin{pmatrix}0\\ \vdots\\ 0\\ \xi_1\\ \vdots\\ \xi_q\end{pmatrix}\right\rangle_{\mathbb{C}^n}\\
&= \lim_{\varepsilon\to 0}\sum_1^q\frac{1}{\lambda_k+\varepsilon}|\xi_k|^2\neq\infty,
\end{aligned}$$

since $\vec{g}\neq 0$. This contradiction gives the proof of the necessity. Conversely, assume det $\mathbb{Q}=0$. With the notation already used above, assume that there is $\eta\neq 0$ such that

$$U\vec{g}=\begin{pmatrix}\vdots\\ \eta\\ \vdots\\ \xi_1\\ \vdots\\ \xi_q\end{pmatrix}.$$

We have

$$\begin{aligned}
|\lim_{\varepsilon\to 0}V_{\vec{\theta}}(-\varepsilon)| &= \lim_{\varepsilon\to 0}\langle(\mathbb{P}^{-1}\mathbb{Q}+\varepsilon I)^{-1}\vec{g}\,,\ \vec{g}\,\rangle_{\mathbb{C}^n_{\mathbb{P}}}\\
&= \langle(U\mathbb{P}^{-1}\mathbb{Q}U^*+\varepsilon I)^{-1}U\vec{g}\,, U\vec{g}\,\rangle_{\mathbb{C}^n_{\mathbb{P}}}\\
&= \lim_{\varepsilon\to 0}\left\langle\operatorname{diag}\left(\frac{1}{\varepsilon},\ldots,\frac{1}{\varepsilon},\frac{1}{\lambda_1+\varepsilon},\ldots,\frac{1}{\lambda_q+\varepsilon}\right)\begin{pmatrix}\vdots\\ \eta\\ \vdots\\ \xi_1\\ \vdots\\ \xi_q\end{pmatrix},\begin{pmatrix}\vdots\\ \eta\\ \vdots\\ \xi_1\\ \vdots\\ \xi_q\end{pmatrix}\right\rangle_{\mathbb{C}^n}
\end{aligned}$$

$$\geqslant \lim_{\varepsilon\to 0} \frac{|\eta|^2}{\varepsilon} = \infty.$$

Applying Theorem 9.2 we see that $\vec{T}$ is extremal. □

Theorem 11.8. *Let $\{z_k\}$ and $\{v_k\}$ be an interpolation data with $z_k \in \mathbb{C}_+$. Suppose that the matrix $\mathbb{Q}$ and $\mathbb{P}$ defined above are strictly positive. Then, the operator $\vec{T}$ in* (11.5) *is α-sectorial and the angle α is given by*

$$\tan\alpha = \langle \mathbb{Q}^{-1}\varphi, \varphi\rangle_{\mathbb{C}^m}, \quad \varphi = \begin{pmatrix} v_1^* \\ \vdots \\ v_n^* \end{pmatrix}. \tag{11.54}$$

Proof. By Theorem 9.2,

$$\begin{aligned}\tan\alpha &= \| V_\theta(0^-)\| \\ &= \lim_{x\to 0^-} \left\langle (\mathbb{P}^{-1}\mathbb{Q} + xI)^{-1}\vec{g}, \vec{g}\right\rangle_{\mathbb{C}^m_{\mathbb{P}}} \\ &= \left\langle \mathbb{Q}^{-1}\mathbb{P}\vec{g}, \vec{g}\right\rangle_{\mathbb{C}^m_{\mathbb{P}}} \\ &= \left\langle \mathbb{Q}^{-1}\mathbb{P}\vec{g}, \mathbb{P}\vec{g}\right\rangle_{\mathbb{C}^m},\end{aligned}$$

which allows us to conclude since $g = \mathbb{P}^{-1}\varphi$. □

Corollary 11.9. *Let $\{z_k\}$ and $\{v_k\}$ be an interpolation data (with $z_k \neq z_j^*$) for which the matrices $\mathbb{P}$ and $\mathbb{Q}$ are strictly positive. Then*

$$\mathbb{Q} \geqslant \frac{1}{\langle \mathbb{Q}^{-1}\varphi, \varphi\rangle_{\mathbb{C}^m}}\varphi\varphi^*. \tag{11.55}$$

Proof. The result follows from Theorems 11.1, 11.8 and inequality (11.2). □

Theorem 11.10. *Let $\{z_k\}$ and $\{v_k\}$ be an interpolation data with $z_k \in \mathbb{C}_+$. Suppose also that the matrix $\mathbb{Q} = ((z_k v_k - z_j^* v_j^*)/(z_k - z_j^*))$ is strictly positive. Assume that the Brodskii–Livsic scattering system*

$$\theta = \begin{pmatrix} T & K & I \\ \mathcal{G} & & \mathbb{C} \end{pmatrix}$$

is an interpolation system in the Nevanlinna–Pick interpolation problem with $\dim \mathcal{G} = n$ *for which the z_k are not in the spectrum of T and such that*

$$\bigvee_{k=1}^{m} (T - z_k I)^{-1} K\mathbb{C} = \mathcal{G}.$$

Then, the main operator T is α-sectorial and the value of the angle α is given by formula (11.54).

Proof. Theorem 11.2 (or the analysis of the first part of the paper) implies that the matrix $\mathbb{P} = ((v_k - v_j^*)/(z_k - z_j^*))$ is strictly positive. The same theorem implies that θ is unitarily equivalent to the system (11.5) and (11.6). Theorem 11.8 implies then that the main operator T is α-sectorial and that the angle α is given by (11.54). $\square$

12. Examples

(1) In this section, we present two illustrative examples. The first one is scalar: We let $z_1 = \eta\mathrm{i}$ and $v_1 = \varepsilon\mathrm{i}$ with ε and η strictly positive numbers. It is readily computed that

$$\mathbb{P} = \frac{\varepsilon}{\eta} > 0, \tag{12.1}$$

$$\mathbb{Q} = 0, \tag{12.2}$$

$$\vec{g} = -\eta\mathrm{i}, \tag{12.3}$$

$$\begin{aligned}\vec{T} &= \mathrm{i}\langle\cdot, \vec{g}\rangle_{\mathbb{C}}\vec{g}\\ &= (\mathrm{i}\eta\varepsilon),\end{aligned} \tag{12.4}$$

$$\vec{A} = 0. \tag{12.5}$$

Hence,

$$V_{\vec{\theta}}(z) = -\frac{\eta\varepsilon}{z}. \tag{12.6}$$

The interpolation condition

$$V_{\vec{\theta}}(\mathrm{i}\eta) = \mathrm{i}\varepsilon \tag{12.7}$$

is clearly satisfied. The operator $\vec{T}$ is extremal. When $\varepsilon \neq 1$, the number $z_1 = \mathrm{i}\eta$ is not an eigenvalue of $\vec{T}$, and the system $\vec{\theta}$ is the unique Brodskii–Livsic system with dimension of the state space equal to 1 and satisfying the interpolation condition.

When $\varepsilon = 1$, Theorem 11.5 allows us still to conclude to uniqueness of the Brodskii–Livsic interpolation system. In this case, $z_1 = \eta\mathrm{i}$ is an eigenvalue of $\vec{T}$.

(2) For the second example, take

$$T = \begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix}.$$

Then,

$$\mathrm{Re}\ T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Furthermore,

$$V_\theta(z) = \langle(\mathrm{Re}\ T - zI)^{-1}g, g\rangle = \frac{z}{1-z^2}.$$

Choose the points $z_1 = 2\mathrm{i}$ and $z_2 = 3\mathrm{i}$. Then,

$$V_\theta(z_1) = \frac{2\mathrm{i}}{5}, \quad V_\theta(z_2) = \frac{3\mathrm{i}}{10}.$$

Thus

$$(\mathrm{Re}\ T - z_1)^{-1}g = \begin{pmatrix} \frac{2\mathrm{i}}{5} \\ \frac{1}{5} \end{pmatrix}, \quad (\mathrm{Re}\ T - z_2 I)^{-1}g = \begin{pmatrix} \frac{3\mathrm{i}}{10} \\ \frac{1}{10} \end{pmatrix}.$$

The eigenvalues of TT^* are equal to $((3 \pm \sqrt{5})/2)$ and thus

$$\min\{|z_1|, |z_2|\} > \|T\|.$$

We are in the setting of the Uniqueness Theorem 8.4, and the colligation is given by

$$\theta = \begin{pmatrix} \begin{pmatrix} \mathrm{i} & 1 \\ 1 & 0 \end{pmatrix} & Kc = c \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} & J = 1 \\ \mathbb{C}^2 & & \mathbb{C} \end{pmatrix}. \tag{12.8}$$

We now compute the *model interpolation solution* presented in Theorem 11.1 of Section 11. We have

$$\mathbb{P} = \begin{pmatrix} \frac{1}{5} & \frac{7}{50} \\ \frac{7}{50} & \frac{1}{10} \end{pmatrix}, \qquad \mathbb{Q} = \begin{pmatrix} 0 & \frac{\mathrm{i}}{50} \\ -\frac{\mathrm{i}}{50} & 0 \end{pmatrix},$$

and thus

$$\overrightarrow{A} = \mathbb{P}^{-1}\mathbb{Q} = \begin{pmatrix} 7\mathrm{i} & 5\mathrm{i} \\ -10\mathrm{i} & -7\mathrm{i} \end{pmatrix}.$$

Furthermore, we have

$$\overrightarrow{g} = \mathbb{P}^{-1}\begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} = \begin{pmatrix} 5\mathrm{i} \\ -10\mathrm{i} \end{pmatrix}.$$

Thus the corresponding colligation is equal to

$$\overrightarrow{\theta} = \begin{pmatrix} \begin{pmatrix} 5\mathrm{i} & \frac{7}{2}\mathrm{i} \\ -6\mathrm{i} & -4\mathrm{i} \end{pmatrix} & \overrightarrow{K}c = c \cdot \begin{pmatrix} 5\mathrm{i} \\ -10\mathrm{i} \end{pmatrix} & J = 1 \\ & & \\ \mathbb{C}^2 \begin{pmatrix} \frac{1}{5} & \frac{7}{50} \\ \frac{7}{50} & \frac{1}{10} \end{pmatrix} & & \mathbb{C} \end{pmatrix}. \tag{12.9}$$

The colligations (12.8) and (12.9) are unitarily equivalent by Remark 8.5.

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